

Neutron β^- -Decay as Laboratory for Test of Standard Model

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We analyse the sensitivity of all experimentally observable asymmetries and energy distributions for the neutron β^- -decay with a polarised neutron and unpolarised decay proton and electron and the lifetime of the neutron to contributions of order 10^{-4} of interactions beyond the Standard model (SM). Since the asymmetries and energy distributions are expressed in terms of the correlation coefficients of the neutron β^- -decay, in order to obtain a theoretical background for the analysis of contributions beyond the SM we revise the calculation of the correlation coefficients within the SM. We take into account a complete set of contributions, induced to next-to-leading order in the large proton mass expansion by the “weak magnetism” and the proton recoil, and the radiative corrections of order (α/π) , calculated to leading order in the large proton mass expansion. We confirm the results, obtained in literature. The contributions of interactions beyond the SM we analyse in the linear approximation with respect to the Herzeg phenomenological coupling constants, introduced at the hadronic level. Such an approximation is good enough for the analysis of contributions of order 10^{-4} of interactions beyond the SM. We show that in such an approximation the correlation coefficients depend only on the axial coupling constant, which absorbs the contributions of the Herzeg left-left and left-right lepton-nucleon current-current interactions (vector and axial-vector interactions beyond the SM), and the Herzeg scalar and tensor coupling constants. In the lifetime of the neutron in addition to the axial coupling constant the contributions of the Herzeg left-left and left-right lepton-nucleon current-current interactions (vector and axial-vector interactions beyond the SM) are absorbed by the Cabibbo-Kobayashi-Maskawa (CKM) matrix element.

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I. INTRODUCTION

In this paper we propose a consistent analysis of the sensitivity of all observable asymmetries and energy distributions of the neutron β^- -decay $n \rightarrow p + e^- + \bar{\nu}_e$ with a polarised neutron and unpolarised decay proton and electron and the lifetime of the neutron [1, 2] to contributions of order 10^{-4} of interactions beyond the Standard model (SM), described at the phenomenological level. Such an order of corrections beyond the SM has been pointed out by Ramsey-Musolf and Su [3] within the Minimal Supersymmetric extension of the SM (MSSM). As has been shown in [4]–[7], at the phenomenological level the neutron β^- -decay may be described by eight complex coupling constants determining the strength of interactions beyond the SM. As a result possible deviations from the SM, causing the contributions of order 10^{-4} to the correlation coefficients of the neutron β^- -decay and the lifetime of the neutron, may be determined in terms of vector, axial-vector, scalar and tensor lepton-baryon weak coupling constants [8].

It is well-known [1, 2] that in the non-relativistic approximation to leading order in the large proton mass expansion, which is equivalent to the leading order of the heavy-baryon approximation, and in the rest frame of the neutron the SM with weak $V - A$ interactions [9] describes the β^- -decay of the neutron in terms of two coupling constants G_V and G_A [1, 2] (see also [10]). The coupling constant G_V is defined by the product $G_V = G_F V_{ud}$ of the Fermi coupling constant $G_F = g_W^2/8M_W^2$ [9], where g_W and M_W are the electroweak coupling constant and the W -boson mass, and V_{ud} is the matrix element of the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix [9]. The coupling constant G_A is equal to $G_A = \lambda G_V$, where $\lambda = G_A/G_V$ is the axial coupling constant, induced by renormalisation

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of the axial hadronic current by strong low-energy interactions [11]. For the weak interactions, invariant under time reversal, the coupling constant λ is real.

The observables of the neutron β^- -decay with unpolarised particles are the lifetime of the neutron τ_n and the correlation coefficient a_0 , describing correlations between 3-momenta of the electron and antineutrino to leading order in the large proton mass expansion in the rest frame of the neutron. Experimentally the correlation coefficient a_0 can be determined, for example, by measuring either the electron-proton energy distribution $a(E_e, T_p)$, where E_e and T_p are the total and kinetic energies of the electron and proton, respectively, or the proton-energy spectrum $a(T_p)$. As a function of the electron energy E_e the correlation coefficient of correlations between the electron and antineutrino 3-momenta we denote as $a(E_e)$.

The neutron β^- -decay with a polarised neutron and unpolarised decay proton and electron is characterised also by two additional observable correlation coefficients A_0 and B_0 , describing to leading order in the large proton mass expansion correlations between the neutron spin and 3-momenta of the electron and antineutrino, respectively. As functions of the electron energy E_e the correlation coefficients of correlations between the neutron spin and 3-momenta of the electron and antineutrino we denote as $A(E_e)$ and $B(E_e)$, respectively. The lifetime of the neutron and the correlation coefficients under consideration are of order $\tau_n \sim 880$ s, $a_0 \sim a(E_e) \sim A_0 \sim A(E_e) \sim -0.1$ and $B_0 \sim B(E_e) \sim 1$ [9]. The coupling constants G_V and λ define the main contributions to the lifetime of the neutron and the correlation coefficients [1, 2] (see also [10]).

However for the description of the neutron β^- -decay at the modern level of experimental accuracies when the experimental data on the axial coupling constant $\lambda = -1.2750(9)$ [1] and the lifetime of the neutron $\tau_n^{(\text{exp})} = 878.5(8)$ s [12] have been obtained with accuracies 0.07 % and 0.09 %, respectively, two coupling constants G_V and λ are not enough for the correct description of the properties of the neutron β^- -decay. Indeed, it is well-known that the radiative corrections to the lifetime of the neutron [13]–[30], calculated to leading order in the large proton mass expansion within the SM with the $V - A$ weak interactions and Quantum Electrodynamics (QED), give the contribution of about 3.9 % [28] (see also [1]). This allows to describe correctly the lifetime of the neutron [10]. The radiative corrections are very important also for the correct determination of the Fermi coupling constant G_F . It may be extracted from the experimental data on the weak coupling constant G_μ of the μ^- -decay $\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$ with the account for the radiative corrections [13]–[16]. The contributions of the radiative corrections to the ratios of the correlation coefficients $a(E_e)/a_0$ and $A(E_e)/A_0$ are by order of magnitude smaller compared with the contribution to the lifetime of the neutron, whereas the correlation coefficient $B(E_e)$ has no radiative corrections to order α/π [31, 32].

Another type of corrections, which may be calculated within the SM and should be taken into account for the description of the neutron β^- -decay on the same footing as the radiative corrections, are the contributions of the “weak magnetism” and the proton recoil [33, 34] (see also [32]), calculated to next-to-leading order in the large proton mass expansion.

The radiative corrections and the corrections from the “weak magnetism” and the proton recoil, calculated to leading order and to next-to-leading order in the large proton mass expansion, respectively, define a complete set of corrections to the observables of the neutron β^- -decay, which should be taken into account within the SM as a background for the analysis of contributions of order 10^{-4} of interactions beyond the SM.

Since the observable asymmetries and energy distributions are defined in terms of the correlation coefficients [1], for the aim of the paper to analyse the sensitivity of the observables of the neutron β^- -decay to contributions of order 10^{-4} of interactions beyond the SM we revise the calculation of the correlation coefficients of the neutron β^- -decay with a polarised neutron and unpolarised decay proton and electron and the lifetime of the neutron within the SM with $V - A$ weak interactions. In addition to the non-relativistic terms a_0 , A_0 and B_0 , calculated to leading order in the large proton mass expansion, we take into account 1) the contributions of the “weak magnetism” and the proton recoil to next-to-leading order in the large M expansion, which provide a complete set of corrections of order $1/M$, where M is an average mass of the neutron and proton $M = (m_n + m_p)/2$, and 2) the radiative corrections of order α/π , calculated to leading order in the large M expansion, where $\alpha = 1/137.036$ is the fine-structure constant [9]. The small parameter of the large M expansion is $E_0/M \sim 10^{-3}$, where $E_0 \sim 1$ MeV is the end-point energy of the electron-energy spectrum. The parameter $E_0/M \sim 10^{-3}$ is commensurable with the parameter of the radiative corrections $\alpha/\pi \sim 10^{-3}$. As we show below due to the strong dependence on the axial coupling constant λ the numerical values of the $1/M$ corrections vary from 10^{-5} to 10^{-1} . In turn the contributions of the radiative corrections do not depend on the axial coupling constant λ . Following Sirin [18] we show that the contributions of the radiative corrections, depending on the axial coupling constant λ , may be absorbed by renormalisation of the Fermi G_F and axial λ coupling constants. The contributions of radiative corrections to the ratios $a(E_e)/a_0$ and $A(E_e)/A_0$ are of order 10^{-3} . The correlation coefficients, calculated within the SM to order $1/M$ and α/π , determine the theoretical background for the analysis of contributions of order 10^{-4} beyond the SM.

A phenomenological analysis of contributions of interactions beyond the SM model shows that these interactions induce only 1) the energy independent contributions and 2) the contributions proportional to m_e/E_e , where m_e is the electron mass. We show below that the contributions of the “weak magnetism” and the proton recoil into the terms

proportional to m_e/E_e of the correlation coefficients $a(E_e)$, $A(E_e)$ and $B(E_e)$ are of order $10^{-3} - 10^{-4}$. Thus, the obtained theoretical expressions for the correlation coefficients should be taken into account for a correct experimental determination of contributions of order 10^{-4} beyond the SM.

The radiative corrections to the correlation coefficients $a(E_e)/a_0$ and $A(E_e)/A_0$ determine the most important and complicated part of the corrections of order 10^{-3} . In the neutron β^- -decay the radiative corrections are defined by 1) the contributions to the continuum-state β^- -decay mode from one-virtual photon exchanges, W -boson and Z -boson exchanges and QCD corrections [24, 27]–[30] and 2) the contribution of the radiative β^- -decay mode $n \rightarrow p + e^- + \bar{\nu}_e + \gamma$ with emission of a real photon [13]–[30] (see also [31, 32]). The sum of the electron-energy and angular distributions of these two decay modes does not suffer from infrared divergences of virtual and real photons.

The radiative corrections to the β^- -decay of the neutron we calculate within the standard finite-photon mass (FPM) regularisation of infrared divergences [13]–[30]. As has been shown by Marciano and Sirlin [22], the FPM regularisation is equivalent to the dimensional regularisation [35].

The radiative corrections to the lifetime of the neutron, calculated within the FPM regularisation, obey the Kinoshita–Lee–Nauenberg (KLN) theorem [36]. According to the KLN theorem, the radiative corrections to the lifetime of the neutron, integrated over the phase volume of the final state of the neutron β^- -decay in the limit of the massless electron $m_e \rightarrow 0$, should not depend on the electron mass m_e . For the first time such an independence of the electron mass in the limit $m_e \rightarrow 0$ has been demonstrated by Kinoshita and Sirlin [15].

We reproduce fully the radiative corrections to the lifetime of the neutron and the correlation coefficients $a(E_e)$ and $A(E_e)$, calculated in [16]–[30] and [31, 32], respectively (see Eq.(7) and Appendix D). We show also that the correlation coefficient $B(E_e)$ has no radiative corrections to order α/π , calculated to leading order in the large M expansion. This agrees well with the results, obtained by Gudkov *et al.* [32]. The corrections of the “weak magnetism” and the proton recoil we calculate in complete agreement with the results, obtained in [32] (see also [34]). Above the background of these corrections the contributions of interactions beyond the SM are calculated to leading order in the large M expansion and in the linear approximation with respect to the Herczeg phenomenological coupling constants, introduced at the hadronic level in terms of the lepton–nucleon current–current interactions (see Appendix G). Such an approximation is good enough for the analysis of the sensitivity of the observables of the neutron β^- -decay to contributions of order 10^{-4} beyond the SM.

The obtained results are applied to the theoretical analysis of 1) the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$, the electron–proton energy distribution $a(E_e, T_p)$, the proton–energy spectrum $a(T_p)$ and the proton recoil asymmetry C_{exp} , used for the experimental determination of the axial coupling constant λ and the correlation coefficients A_0 , B_0 , a_0 and $C_0 = -x_C(A_0 + B_0)$, where $x_C = 0.27491$ is a theoretical numerical factor (see section VII), respectively, 2) the lifetime of the neutron and 3) the sensitivity of the electron–proton energy distribution $a(E_e, T_p)$, the proton–energy spectrum $a(T_p)$, the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$ and C_{exp} and the lifetime of the neutron τ_n to contributions of order 10^{-4} of interactions beyond the SM. The experimental analysis of the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$, the proton–energy spectrum $a(T_p)$ and the proton recoil asymmetry C_{exp} has been carried out in [1, 37], [38, 39], [40] (see also [41]) and [42], respectively.

II. TABLE OF CONTENTS

The paper is organised as follows. In section III we calculate the electron–energy and angular distribution of the neutron β^- -decay with a polarised neutron and unpolarised decay proton and electron. We take into account a complete set of corrections of order $1/M$, caused by the “weak magnetism” and the proton recoil, and the radiative corrections of order α/π . Then we analyse 1) in section IV the electron asymmetry $A_{\text{exp}}(E_e)$, which has been used in [1, 37] for the experimental determination of the axial coupling constant $\lambda = -1.2750(9)$ and the correlation coefficient $A_0^{(\text{exp})} = -0.11933(34)$, 2) in section V the antineutrino asymmetry $B_{\text{exp}}(E_e)$, which has been used for the experimental determination of the correlation coefficient $B_0^{(\text{exp})} = 0.9802(50)$ in [38, 39], 3) in section VI the electron–proton energy distribution $a(E_e, T_p)$ and the proton–energy spectrum $a(T_p)$, which may be used for the experimental determination of the axial coupling constant λ and the correlation coefficient a_0 , 4) in section VII the proton recoil asymmetry C_{exp} , which has been used for the experimental determination of the correlation coefficient $C_0^{(\text{exp})} = -x_C(A_0 + B_0) = -0.2377(26)$ [42], and 5) in section VIII the lifetime of the neutron. In section IX we propose the theoretical analysis of the sensitivity of the electron–proton energy distribution $a(E_e, T_p)$, the proton–energy spectrum $a(T_p)$, the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$ and C_{exp} and the lifetime of the neutron to contributions of order 10^{-4} of interactions beyond the SM. We show that to linear approximation with respect to the Herczeg phenomenological coupling constants the Herczeg coupling constants a_{LL}^h and a_{LR}^h of the left–left and left–right lepton–nucleon current–current interactions (vector and axial–vector interactions beyond the SM) can be fully absorbed by the axial coupling constants and cannot be determined from the experimental data on the electron–proton energy

distribution $a(E_e, T_p)$, the proton-energy spectrum $a(T_p)$ and the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$ and $C_{\text{exp}}(E_e)$. This agrees well with the results obtained in [43]–[45]. In the lifetime of the neutron the contributions of the Herzeg left–left and left–right lepton–nucleon current–current interactions (vector and axial–vector interactions beyond the SM) become unobservable after the renormalisation of the CKM matrix element $V_{ud} \rightarrow (V_{ud})_{\text{eff}} = V_{ud}(1 + a_{LL}^h + a_{LR}^h)$ (see section IX) in agreement with [43]–[45]. In the conclusion X we summarised the obtained results. In Appendix A we give a detailed calculation of the amplitude, the electron-energy and angular distribution of the continuum-state β^- -decay of the neutron to next-to-leading order in the large M expansion. In Appendix B we calculate the rate, the electron-energy and angular distribution of the radiative β^- -decay of the neutron with a polarised neutron and unpolarised decay particles and their contributions to the rate, the electron-energy and angular distribution of the neutron β^- -decay. We compare our results for the branching ratios of the radiative β^- -decay of the neutron with the experimental data, obtained by Nico *et al.* [46] and Cooper *et al.* [47] (see also [48]), and the theoretical results, obtained in [49, 50]. In Appendix C within the FPM regularisation [13]–[30] we analyse the infrared divergent contributions of one-virtual photon exchanges in the continuum-state β^- -decay mode of the neutron. In Appendix D we give a detailed calculation of the radiative corrections, caused by one-virtual photon exchanges in the continuum-state β^- -decay of the neutron and the radiative β^- -decay of the neutron. We define the radiative corrections to the lifetime of the neutron and the correlation coefficients by the functions $g_n(E_e)$ and $f_n(E_e)$, respectively. We show also that in the rest frame of the neutron and in the non-relativistic approximation for the proton one-virtual photon exchanges induce effective scalar and tensor weak lepton–baryon couplings of order $\alpha G_F/\pi$ and $\alpha \lambda G_F/\pi$, respectively, depending on the electron energy E_e . In Appendix E we show that the radiative corrections to the lifetime of the neutron, defined by the function $g_n(E_e)$, satisfy the KLN theorem. This agrees with the analysis of the radiative corrections, carried out in [15]. In Appendix F we calculate the radiative corrections to the lifetime of the neutron, using the procedure proposed by Sirlin [18]. We confirm that the unambiguity of the observable radiative corrections to the lifetime of the neutron is caused by gauge invariance of the amplitude of one-virtual photon exchanges in the continuum-state β^- -decay of the neutron [18]. In Appendix G we calculate the contributions to the correlation coefficients and the lifetime of the neutron from phenomenological vector, axial–vector, scalar and tensor interactions beyond the SM [4]–[7]. In Appendix H we calculate the proton recoil corrections, caused by the electron–proton Coulomb interaction in the final state of the neutron β^- -decay. We show that in the experimentally used electron energy region $250 \text{ keV} \leq T_e = E_e - m_e \leq 455 \text{ keV}$ [1] the contributions of these corrections to the correlation coefficients are of order $10^{-6} - 10^{-5}$ and may be neglected for the analysis of contributions of order 10^{-4} . In Appendix I we give a detailed calculation of the electron–proton energy–momentum and angular distribution of the neutron β^- -decay with a polarised neutron and unpolarised decay proton and electron. We calculate the electron–proton energy distribution $a(E_e, T_p)$, the proton-energy spectrum $a(T_p)$ and the correlation coefficient C of the proton recoil asymmetry by taking into account the $1/M$ corrections from the “weak magnetism” and the proton recoil and the radiative corrections of order α/π , calculated to leading order in the large M expansion.

III. ELECTRON-ENERGY AND ANGULAR DISTRIBUTION OF NEUTRON β^- -DECAY IN STANDARD MODEL

For the analysis of the electron-energy and angular distribution of the continuum-state β^- -decay of the neutron we use the Hamiltonian of $V - A$ interactions with a real axial coupling constant λ and the contribution of the “weak magnetism” [35, 51]

$$\mathcal{H}_W(x) = \frac{G_F}{\sqrt{2}} V_{ud} \left\{ [\bar{\psi}_p(x) \gamma_\mu (1 + \lambda \gamma^5) \psi_n(x)] + \frac{\kappa}{2M} \partial^\nu [\bar{\psi}_p(x) \sigma_{\mu\nu} \psi_n(x)] \right\} [\bar{\psi}_e(x) \gamma^\mu (1 - \gamma^5) \psi_\nu(x)] \quad (1)$$

invariant under time reversal, where $\psi_p(x)$, $\psi_n(x)$, $\psi_e(x)$ and $\psi_\nu(x)$ are the field operators of the proton, neutron, electron and antineutrino, respectively, γ^μ , γ^5 and $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$ are the Dirac matrices [35] and $\kappa = \kappa_p - \kappa_n = 3.7058$ is the isovector anomalous magnetic moment of the nucleon, defined by the anomalous magnetic moments of the proton $\kappa_p = 1.7928$ and the neutron $\kappa_n = -1.9130$ and measured in nuclear magneton [9].

For numerical calculations we use $G_F = 1.1664 \times 10^{-11} \text{ MeV}^{-2}$ and $|V_{ud}| = 0.97428(15)$ [9]. The value of the CKM matrix element $|V_{ud}| = 0.97428(15)$ agrees well with $|V_{ud}| = 0.97425(22)$, measured from the superallowed $0^+ \rightarrow 0^+$ nuclear β^- -decays [52]. It satisfies also well the unitarity condition $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.99999(43)$ for the CKM matrix elements [9]. The error $\Delta_U = \pm 0.00043$ of the unitarity condition is determined by the errors of the CKM matrix elements $|V_{ud}| = 0.97428 \pm 0.00015$, $|V_{us}| = 0.2253 \pm 0.0007$ and $|V_{ub}| = 0.00347_{-0.00012}^{+0.00016}$ (see Eq.(11.27) in p.150 of Ref.[9]). As a result the error of the unitarity condition is equal to

$$\Delta_U = \sqrt{\sum_q |2V_{uq} \Delta V_{uq}|^2} = 0.00043,$$

where $q = d, s, b$ and $\Delta V_{ud} = 0.00015$, $\Delta V_{us} = 0.0007$ and $\Delta V_{ub} = 0.00016$, respectively.

The amplitude of the continuum-state β^- -decay of the neutron, calculated in the rest frame of the neutron and to next-to-leading order in the large M expansion taking into account the contributions of the “weak magnetism” and the proton recoil, is (see Appendix A)

$$M(n \rightarrow p e^- \bar{\nu}_e) = -2m_n \frac{G_F}{\sqrt{2}} V_{ud} \left\{ [\varphi_p^\dagger \varphi_n] [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - \tilde{\lambda} [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] \right. \\ \left. - \frac{m_e}{2M} [\varphi_p^\dagger \varphi_n] [\bar{u}_e (1 - \gamma^5) v_{\bar{\nu}}] + \frac{\tilde{\lambda}}{2M} [\varphi_p^\dagger (\vec{\sigma} \cdot \vec{k}_p) \varphi_n] [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - i \frac{\kappa + 1}{2M} [\varphi_p^\dagger (\vec{\sigma} \times \vec{k}_p) \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] \right\}, \quad (2)$$

where φ_p and φ_n are the Pauli spinorial wave functions of the proton and neutron and u_e and $v_{\bar{\nu}}$ are the Dirac bispinor wave functions of the electron and antineutrino, respectively. Then, \vec{k}_p is a 3-momentum of the proton related to 3-momenta of the electron \vec{k}_e and antineutrino \vec{k} as $\vec{k}_p = -\vec{k}_e - \vec{k}$, $\tilde{\lambda} = \lambda(1 - E_0/2M)$, where $E_0 = (m_n^2 - m_p^2 + m_e^2)/2m_n = 1.2927$ MeV is the end-point energy of the electron-energy spectrum, calculated for $m_n = 939.5654$ MeV, $m_p = 938.2720$ MeV and $m_e = 0.5110$ MeV [9]. From Eq.(2) one may see that the parameter of the large M expansion or the $1/M$ corrections to the amplitude of the β^- -decay of the neutron is $k_p/M \sim E_0/M \sim 10^{-3}$. The detailed calculation of the amplitude Eq.(2) is given in Appendix A.

The electron-energy and angular distribution of the neutron β^- -decay takes the form [1, 32]

$$\frac{d^5 \lambda_n(E_e, \vec{k}_e, \vec{k}, \vec{\xi}_n)}{dE_e d\Omega_e d\Omega} = (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{32\pi^5} (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) \Phi_{\beta_c^-}(\vec{k}_e, \vec{k}) \tilde{\zeta}(E_e) \\ \times \left(1 + \tilde{a}(E_e) \frac{\vec{k}_e \cdot \vec{k}}{E_e E} + \tilde{A}(E_e) \frac{\vec{\xi}_n \cdot \vec{k}_e}{E_e} + \tilde{B}(E_e) \frac{\vec{\xi}_n \cdot \vec{k}}{E} + \tilde{K}_n(E_e) \frac{(\vec{\xi}_n \cdot \vec{k}_e)(\vec{k}_e \cdot \vec{k})}{E_e^2 E} + \tilde{Q}_n(E_e) \frac{(\vec{\xi}_n \cdot \vec{k})(\vec{k}_e \cdot \vec{k})}{E_e E^2} \right. \\ \left. + \tilde{D}(E_e) \frac{\vec{\xi}_n \cdot (\vec{k}_e \times \vec{k})}{E_e E} \right), \quad (3)$$

where $E = E_0 - E_e$ is the antineutrino energy and $d\Omega_e$ and $d\Omega$ are the infinitesimal elements of the solid angles of the electron and antineutrino 3-momenta relative to the neutron spin, respectively.

The function $\Phi_{\beta_c^-}(\vec{k}_e, \vec{k})$ is defined by the contribution of the proton recoil. To next-to-leading order in the large M expansion it is equal to (see Eqs.(A-19) - (A-23) in Appendix A)

$$\Phi_{\beta_c^-}(\vec{k}_e, \vec{k}) = 1 + \frac{3}{M} \left(E_e - \frac{\vec{k}_e \cdot \vec{k}}{E} \right). \quad (4)$$

The function $F(E_e, Z = 1)$ is the relativistic Fermi function [53, 54] (see also [34])

$$F(E_e, Z = 1) = \left(1 + \frac{1}{2}\gamma \right) \frac{4(2r_p m_e \beta)^{2\gamma}}{\Gamma^2(3 + 2\gamma)} \frac{e^{\pi\alpha/\beta}}{(1 - \beta^2)^\gamma} \left| \Gamma \left(1 + \gamma + i \frac{\alpha}{\beta} \right) \right|^2, \quad (5)$$

where $\beta = k_e/E_e = \sqrt{E_e^2 - m_e^2}/E_e$ is the electron velocity, $\gamma = \sqrt{1 - \alpha^2} - 1$, r_p is the electric radius of the proton and $\alpha = 1/137.036$ is the fine-structure constant. In numerical calculations we will use $r_p = 0.841$ fm [55].

Following [32] we transcribe the r.h.s. of Eq.(3) into the form

$$\frac{d^5 \lambda_n(E_e, \vec{k}_e, \vec{k}, \vec{\xi}_n)}{dE_e d\Omega_e d\Omega} = (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{32\pi^5} (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) \zeta(E_e) \\ \times \left\{ 1 + a(E_e) \frac{\vec{k}_e \cdot \vec{k}}{E_e E} + A(E_e) \frac{\vec{\xi}_n \cdot \vec{k}_e}{E_e} + B(E_e) \frac{\vec{\xi}_n \cdot \vec{k}}{E} + K_n(E_e) \frac{(\vec{\xi}_n \cdot \vec{k}_e)(\vec{k}_e \cdot \vec{k})}{E_e^2 E} + Q_n(E_e) \frac{(\vec{\xi}_n \cdot \vec{k})(\vec{k}_e \cdot \vec{k})}{E_e E^2} \right. \\ \left. + D(E_e) \frac{\vec{\xi}_n \cdot (\vec{k}_e \times \vec{k})}{E_e E} - 3 \frac{E_e}{M} \frac{1 - \lambda^2}{1 + 3\lambda^2} \left(\frac{(\vec{k}_e \cdot \vec{k})^2}{E_e^2 E^2} - \frac{1}{3} \frac{k_e^2}{E_e^2} \right) \right\}. \quad (6)$$

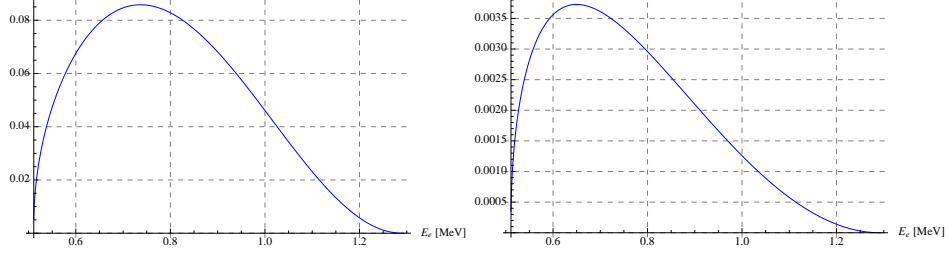


FIG. 1: The densities $(\alpha/\pi) g_n(E_e) \rho_{\beta_c}^-(E_e)$ (left) and $(\alpha/\pi) f_n(E_e) \rho_{\beta_c}^-(E_e)$ (right), measured in MeV^{-1} , of the radiative corrections to the lifetime of the neutron and the correlation coefficients $a(E_e)$ and $A(E_e)$, where $\rho_{\beta_c}^-(E_e)$ is the electron-energy spectrum density Eq.(D-59).

The correlation coefficients are given by (see Appendix A, B, C and D)

$$\begin{aligned}
\zeta(E_e) &= \left(1 + \frac{\alpha}{\pi} g_n(E_e)\right) + \frac{1}{M} \frac{1}{1+3\lambda^2} \left[-2\lambda(\lambda - (\kappa+1)) E_0 + (10\lambda^2 - 4(\kappa+1)\lambda + 2) E_e \right. \\
&\quad \left. - 2\lambda(\lambda - (\kappa+1)) \frac{m_e^2}{E_e} \right], \\
\zeta(E_e) a(E_e) &= a_0 \left(1 + \frac{\alpha}{\pi} g_n(E_e) + \frac{\alpha}{\pi} f_n(E_e)\right) + \frac{1}{M} \frac{1}{1+3\lambda^2} \left[2\lambda(\lambda - (\kappa+1)) E_0 - 4\lambda(3\lambda - (\kappa+1)) E_e \right], \\
\zeta(E_e) A(E_e) &= A_0 \left(1 + \frac{\alpha}{\pi} g_n(E_e) + \frac{\alpha}{\pi} f_n(E_e)\right) + \frac{1}{M} \frac{1}{1+3\lambda^2} \left[(\lambda^2 - \kappa\lambda - (\kappa+1)) E_0 \right. \\
&\quad \left. - (5\lambda^2 - (3\kappa-4)\lambda - (\kappa+1)) E_e \right], \\
\zeta(E_e) B(E_e) &= B_0 \left(1 + \frac{\alpha}{\pi} g_n(E_e)\right) + \frac{1}{M} \frac{1}{1+3\lambda^2} \left[-2\lambda(\lambda - (\kappa+1)) E_0 \right. \\
&\quad \left. + (7\lambda^2 - (3\kappa+8)\lambda + (\kappa+1)) E_e - (\lambda^2 - (\kappa+2)\lambda + (\kappa+1)) \frac{m_e^2}{E_e} \right], \\
\zeta(E_e) K_n(E_e) &= \frac{1}{M} \frac{1}{1+3\lambda^2} (5\lambda^2 - (\kappa-4)\lambda - (\kappa+1)) E_e, \\
\zeta(E_e) Q_n(E_e) &= \frac{1}{M} \frac{1}{1+3\lambda^2} \left[(\lambda^2 - (\kappa+2)\lambda + (\kappa+1)) E_0 - (7\lambda^2 - (\kappa+8)\lambda + (\kappa+1)) E_e \right], \\
\zeta(E_e) D(E_e) &= 0,
\end{aligned} \tag{7}$$

where the correlation coefficients a_0 , A_0 and B_0 are determined by [1] (see also [10])

$$a_0 = \frac{1-\lambda^2}{1+3\lambda^2}, \quad A_0 = -2 \frac{\lambda(1+\lambda)}{1+3\lambda^2}, \quad B_0 = -2 \frac{\lambda(1-\lambda)}{1+3\lambda^2}. \tag{8}$$

The radiative corrections are determined by the functions $g_n(E_e)$ and $f_n(E_e)$, which are given in Eq.(D-58) of Appendix D.

The functions $(\alpha/\pi) g_n(E_e)$ and $(\alpha/\pi) f_n(E_e)$ describe the radiative corrections to the lifetime of the neutron and the correlation coefficients $a(E_e)$ and $A(E_e)$, respectively. They are equal to the radiative corrections, calculated in [18]–[30] and [31, 32], respectively. We show in Appendix D that the correlation coefficient $B(E_e)$ has no radiative corrections to order α/π . This agrees also well with the results, obtained in [32]. The densities of the radiative corrections $(\alpha/\pi) g_n(E_e) \rho_{\beta_c}^-(E_e)$ and $(\alpha/\pi) f_n(E_e) \rho_{\beta_c}^-(E_e)$, where $\rho_{\beta_c}^-(E_e)$ is the electron-energy spectrum density Eq.(D-59), are plotted in Fig. 1.

The coefficients $K_n(E_e)$ and $Q_n(E_e)$ have been introduced in [31, 32] and calculated within the effective quantum field theory, based on the heavy-baryon Chiral Perturbation Theory (HBChPT). Our results of the calculation of the correlation coefficients, carried out to next-to-leading order in the large M expansion (see Appendix A), agree fully with the expressions, calculated in [32].

The correlation coefficient $D(E_e)$ relates to a violation of time reversal invariance. The correlation coefficient $D(E_e)$, calculated for weak interactions, described by the Hamiltonian Eq.(1) with a real axial coupling constant λ , is equal to zero $D(E_e) = 0$. In the SM a non-vanishing correlation coefficient $D(E_e)$ may appear due to long-range [56]–[59] (see also [33]) and short-range [60] mechanisms of time reversal violation. In the long-range mechanism of time

reversal violation the correlation coefficient $D(E_e)$ is induced by the electron–proton interaction in the final–state of the decay due to the distortion of the electron wave function in the Coulomb field of the proton [53, 54], the “weak magnetism” and the proton recoil. In the short–range mechanism of time reversal violation the correlation coefficient $D(E_e)$ takes a contribution from the CP–violating phase δ of the CKM quark mixing matrix [9]. According to [60], the contribution of the long–range mechanism of time reversal violation dominates by many orders of magnitude in comparison with the contribution of the short–range one. As has been shown in [56]–[59] the correlation coefficient $D(E_e)$ is a function of the electron energy E_e . Using the results obtained in [56], for the electron kinetic energies $250 \text{ keV} \leq T_e \leq 455 \text{ keV}$ [1, 37, 39] and the axial coupling constant $\lambda = -1.2750$ we obtain that $D(E_e) \sim 10^{-5}$. Hence the contribution of the long–range mechanism of time reversal violation to the correlation coefficient $D(E_e)$ is smaller compared with contributions of order 10^{-4} , which may be induced by interactions beyond the SM. Recently the correlation coefficient $D(E_e)$ has been calculated within heavy–baryon effective field theory by Ando *et al.* [61]. The authors have reproduced the result, obtained by Callan and Treiman [56], and have found a correction, which is smaller compared with 10^{-7} in the experimental region of electron kinetic energies $250 \text{ keV} \leq T_e \leq 455 \text{ keV}$ [1, 37, 39].

From Eq.(7) the correlation coefficients under consideration, taking into account the contributions of order $1/M$ and α/π , we define as follows

$$\begin{aligned}
a(E_e) &= a_0 \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) + \frac{1}{M} \left[\frac{2\lambda(\lambda - (\kappa + 1))}{1 + 3\lambda^2} E_0 - \frac{4\lambda(3\lambda - (\kappa + 1))}{1 + 3\lambda^2} E_e \right] - a_0 \delta\zeta(E_e), \\
A(E_e) &= A_0 \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) + \frac{1}{M} \left[\frac{\lambda^2 - \kappa\lambda - (\kappa + 1)}{1 + 3\lambda^2} E_0 - \frac{5\lambda^2 - (3\kappa - 4)\lambda - (\kappa + 1)}{1 + 3\lambda^2} E_e \right] - A_0 \delta\zeta(E_e), \\
B(E_e) &= B_0 + \frac{1}{M} \left[-\frac{2\lambda(\lambda - (\kappa + 1))}{1 + 3\lambda^2} E_0 + \frac{7\lambda^2 - (3\kappa + 8)\lambda + (\kappa + 1)}{1 + 3\lambda^2} E_e - \frac{\lambda^2 - (\kappa + 2)\lambda + (\kappa + 1)}{1 + 3\lambda^2} \frac{m_e^2}{E_e} \right] \\
&\quad - B_0 \delta\zeta(E_e), \\
K_n(E_e) &= \frac{1}{M} \frac{5\lambda^2 - (\kappa - 4)\lambda - (\kappa + 1)}{1 + 3\lambda^2} E_e, \\
Q_n(E_e) &= \frac{1}{M} \left[\frac{\lambda^2 - (\kappa + 2)\lambda + (\kappa + 1)}{1 + 3\lambda^2} E_0 - \frac{7\lambda^2 - (\kappa + 8)\lambda + (\kappa + 1)}{1 + 3\lambda^2} E_e \right].
\end{aligned} \tag{9}$$

Using the following expansion

$$\begin{aligned}
\frac{1}{\zeta(E_e)} &= \left(1 - \frac{\alpha}{\pi} g_n(E_e)\right) - \delta\zeta(E_e), \\
\delta\zeta(E_e) &= \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left(\zeta_1 E_0 + \zeta_2 E_e + \zeta_3 \frac{m_e^2}{E_e} \right), \\
\zeta_1 &= -2\lambda(\lambda - (\kappa + 1)), \\
\zeta_2 &= 10\lambda^2 - 4(\kappa + 1)\lambda + 2, \\
\zeta_3 &= -2\lambda(\lambda - (\kappa + 1))
\end{aligned} \tag{10}$$

we transcribe the correlation coefficients $a(E_e)$, $A(E_e)$ and $B(E_e)$ into the form, which is similar to that proposed by Wilkinson for the correlation coefficient $A^{(W)}(E_e)$ (see Eq.(20)). We get

$$\begin{aligned}
a(E_e) &= a_0 \left\{ 1 + \frac{1}{M} \frac{1}{(1 - \lambda^2)(1 + 3\lambda^2)} \left(a_1 E_0 + a_2 E_e + a_3 \frac{m_e^2}{E_e} \right) \right\} \left(1 + \frac{\alpha}{\pi} f_n(E_e) \right), \\
a_1 &= 4\lambda(\lambda^2 + 1)(\lambda - (\kappa + 1)), \\
a_2 &= -26\lambda^4 + 8(\kappa + 1)\lambda^3 - 20\lambda^2 + 8(\kappa + 1)\lambda - 2, \\
a_3 &= -2\lambda(\lambda^2 - 1)(\lambda - (\kappa + 1))
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
A(E_e) &= A_0 \left\{ 1 - \frac{1}{M} \frac{1}{2\lambda(1 + \lambda)(1 + 3\lambda^2)} \left(A_1 E_0 + A_2 E_e + A_3 \frac{m_e^2}{E_e} \right) \right\} \left(1 + \frac{\alpha}{\pi} f_n(E_e) \right), \\
A_1 &= -\lambda^4 + \kappa\lambda^3 + (\kappa + 2)\lambda^2 - \kappa\lambda - (\kappa + 1), \\
A_2 &= 5\lambda^4 + \kappa\lambda^3 - (5\kappa + 6)\lambda^2 + 3\kappa\lambda + (\kappa + 1), \\
A_3 &= -4\lambda^2(\lambda + 1)(\lambda - (\kappa + 1))
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
B(E_e) &= B_0 \left\{ 1 - \frac{1}{M} \frac{1}{2\lambda(1-\lambda)(1+3g_A^2)} \left(B_1 E_0 + B_2 E_e + B_3 \frac{m_e^2}{E_e} \right) \right\}, \\
B_1 &= -2\lambda(\lambda+1)^2(\lambda-(\kappa+1)), \\
B_2 &= \lambda^4 - (\kappa-4)\lambda^3 - (5\kappa+2)\lambda^2 - (3\kappa+4)\lambda + (\kappa+1), \\
B_3 &= (\lambda^2-1)(\lambda-1)(\lambda-(\kappa+1)).
\end{aligned} \tag{13}$$

For the derivation of Eq.(11), Eq.(12) and Eq.(13) we have neglected the terms of order $(\alpha/\pi)(E_0/M) \sim 10^{-6}$, which are smaller compared with contributions of order 10^{-4} beyond the SM.

In order to estimate the values of the obtained $1/M$ corrections we calculate them at $\lambda = -1.2750$. This gives

$$\begin{aligned}
\delta\zeta(E_e) &= \frac{1}{M} \frac{1}{1+3\lambda^2} \left(\zeta_1 E_0 + \zeta_2 E_e + \zeta_3 \frac{m_e^2}{E_e} \right) = -3.57 \times 10^{-3} + 9.90 \times 10^{-3} \frac{E_e}{E_0} - 1.41 \times 10^{-3} \frac{m_e}{E_e}, \\
\delta a(E_e) &= \frac{1}{M} \frac{1}{(1-\lambda^2)(1+3\lambda^2)} \left(a_1 E_0 + a_2 E_e + a_3 \frac{m_e^2}{E_e} \right) = -3.00 \times 10^{-2} + 8.59 \times 10^{-2} \frac{E_e}{E_0} + 1.41 \times 10^{-3} \frac{m_e}{E_e}, \\
\delta A(E_e) &= -\frac{1}{M} \frac{1}{2\lambda(1+\lambda)(1+3\lambda^2)} \left(A_1 E_0 + A_2 E_e + A_3 \frac{m_e^2}{E_e} \right) = 3.44 \times 10^{-4} + 1.47 \times 10^{-2} \frac{E_e}{E_0} + 1.42 \times 10^{-3} \frac{m_e}{E_e}, \\
\delta B(E_e) &= -\frac{1}{M} \frac{1}{2\lambda(1-\lambda)(1+3\lambda^2)} \left(B_1 E_0 + B_2 E_e + B_3 \frac{m_e^2}{E_e} \right) = -4.66 \times 10^{-5} - 2.97 \times 10^{-3} \frac{E_e}{E_0} + 1.36 \times 10^{-4} \frac{m_e}{E_e}, \\
K_n(E_e) &= 7.14 \times 10^{-4} \frac{E_e}{E_0}, \\
Q_n(E_e) &= 3.19 \times 10^{-3} - 7.27 \times 10^{-3} \frac{E_e}{E_0}.
\end{aligned} \tag{14}$$

Due to strong dependence on the axial coupling constant λ the numerical values of the contributions of the “weak magnetism” and the proton recoil, calculated at $\lambda = -1.2750$, vary from 10^{-5} to 10^{-1} for energy independent and energy dependent terms.

Summary. The correlation coefficients of the electron–energy and angular distribution of the neutron β^- –decay with a polarised neutron and unpolarised decay proton and electron are calculated by taking into account a complete set of the $1/M$ corrections, caused by the “weak magnetism” and the proton recoil, and the radiative corrections of order α/π , calculated to leading order in the large M expansion. The obtained expressions for the correlation coefficients should be used as a theoretical background for contributions of order 10^{-4} of interactions beyond the SM, calculated in Appendix G (see Eq.(G-8)). Contributions of order 10^{-4} of interactions beyond the SM may be determined by measuring the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$ and C_{exp} between the neutron spin and the 3–momenta of the decay particles, the electron–proton energy distribution $a(E_e, T_p)$ and the proton–energy spectrum $a(T_p)$, related to correlations between the 3–momenta of the electron and proton, and the lifetime of the neutron τ_n (see section IX).

IV. STANDARD MODEL ANALYSIS OF EXPERIMENTAL DETERMINATION OF CORRELATION COEFFICIENT A_0 . ELECTRON ASYMMETRY $A_{\text{exp}}(E_e)$

For the experimental determination of the correlation coefficient A_0 , defining correlations between the neutron spin and the electron 3–momentum in the SM to leading order in the large M expansion [1], the directions of the emission of the antineutrino are not fixed and one has to integrate over the antineutrino 3–momentum \vec{k} . As a result we arrive at the following electron–energy and angular distribution [1, 37]

$$\begin{aligned}
\frac{d^2 \lambda_n(E_e, \vec{k}_e, \vec{\xi}_n)}{dE_e d\Omega_e} &= (1+3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{8\pi^4} (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z=1) \zeta(E_e) \\
&\times \left(1 + A^{(W)}(E_e) \left(1 + \frac{\alpha}{\pi} f_n(E_e) \right) \vec{\xi}_n \cdot \vec{\beta} \right),
\end{aligned} \tag{15}$$

where we have denoted

$$A(E_e) + \frac{1}{3} Q_n(E_e) = A^{(W)}(E_e) \left(1 + \frac{\alpha}{\pi} f_n(E_e) \right), \tag{16}$$

$d\Omega_e = 2\pi \sin \theta_e d\theta_e$ is an infinitesimal solid angle of the electron 3-momentum with respect to the neutron spin and $\vec{\xi}_n \cdot \vec{\beta} = P\beta \cos \theta_e$ with the neutron polarisation $P = |\vec{\xi}_n| \leq 1$. The correlation coefficients $A(E_e)$ and $Q_n(E_e)$ are given in Eq.(12) and Eq.(9), respectively. The contribution, proportional to $Q_n(E_e)$, with structure $\vec{\xi}_n \cdot \vec{k}_e/E_e$ we obtain having integrated the term with the structure $(\vec{\xi}_n \cdot \vec{k})(\vec{k}_e \cdot \vec{k})/E_e E^2$ in Eq.(6) over directions of the antineutrino 3-momentum \vec{k} .

The asymmetry, which may be used for the experimental determination of the axial coupling constant λ and the correlation coefficient A_0 , takes the form

$$A_{\text{exp}}(E_e) = \frac{N^+(E_e) - N^-(E_e)}{N^+(E_e) + N^-(E_e)} = \frac{1}{2} A^{(W)}(E_e) \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) P\beta (\cos \theta_1 + \cos \theta_2), \quad (17)$$

where $N^\pm(E_e)$ are the numbers of events of the emission of the electron forward (+) and backward (−) with respect to the neutron spin into the solid angle $\Delta\Omega_{12} = 2\pi(\cos \theta_1 - \cos \theta_2)$ with $0 \leq \varphi \leq 2\pi$ and $\theta_1 \leq \theta_e \leq \theta_2$. They are determined by [62]

$$\begin{aligned} N^+(E_e) &= 2\pi N(E_e) \int_{\theta_1}^{\theta_2} \left(1 + A^{(W)}(E_e) \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) P\beta \cos \theta_e\right) \sin \theta_e d\theta_e = \\ &= 2\pi N(E_e) \left(1 + \frac{1}{2} A^{(W)}(E_e) \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) P\beta (\cos \theta_1 + \cos \theta_2)\right) (\cos \theta_1 - \cos \theta_2), \\ N^-(E_e) &= 2\pi N(E_e) \int_{\pi-\theta_1}^{\pi-\theta_2} \left(1 + A^{(W)}(E_e) \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) P\beta \cos \theta_e\right) \sin \theta_e d\theta_e = \\ &= 2\pi N(E_e) \left(1 - \frac{1}{2} A^{(W)}(E_e) \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) P\beta (\cos \theta_1 + \cos \theta_2)\right) (\cos \theta_1 - \cos \theta_2), \end{aligned} \quad (18)$$

where $N(E_e)$ is the normalisation factor equal to

$$N(E_e) = (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{8\pi^4} (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z=1) \zeta(E_e). \quad (19)$$

The correlation coefficient $A^{(W)}(E_e)$ is

$$\begin{aligned} A^{(W)}(E_e) &= A_0 \left\{ 1 - \frac{1}{M} \frac{1}{2\lambda(1+\lambda)(1+3\lambda^2)} \left(A_1^{(W)} E_0 + A_2^{(W)} E_e + A_3^{(W)} \frac{m_e^2}{E_e} \right) \right\}, \\ A_1^{(W)} &= \frac{2}{3} \left(-3\lambda^3 + (3\kappa + 5)\lambda^2 - (2\kappa + 1)\lambda - (\kappa + 1) \right) = -2 \left(\lambda - (\kappa + 1) \right) \left(\lambda^2 - \frac{2}{3}\lambda - \frac{1}{3} \right), \\ A_2^{(W)} &= \frac{2}{3} \left(-3\lambda^4 + (3\kappa + 12)\lambda^3 - (9\kappa + 14)\lambda^2 + (5\kappa + 4)\lambda + (\kappa + 1) \right) = -2 \left(\lambda - (\kappa + 1) \right) \left(\lambda^3 - 3\lambda^2 + \frac{5}{3}\lambda + \frac{1}{3} \right), \\ A_3^{(W)} &= -4\lambda^2(\lambda + 1) \left(\lambda - (\kappa + 1) \right). \end{aligned} \quad (20)$$

It agrees well with the result, obtained by Wilkinson [34, 63]. We note that the correlation coefficient $A(E_e) + \frac{1}{3}Q_n(E_e)$ differs from the Wilkinson correlation coefficient $A^{(W)}(E_e)$ by the contribution of the radiative corrections, described by the function $(\alpha/\pi) f_n(E_e)$. In the replacement $A(E_e) + \frac{1}{3}Q_n(E_e) \rightarrow A^{(W)}(E_e)(1 + (\alpha/\pi) f_n(E_e))$ we have neglected the contributions of order $(\alpha/\pi)(E_0/M) \sim 10^{-6}$, which are smaller compared with contributions of order 10^{-4} of our interest. The contribution to the correlation coefficient $A^{(W)}(E_e)$ of the “weak magnetism” and the proton recoil, calculated at $\lambda = -1.2750$, is equal to

$$\begin{aligned} \delta A^{(W)}(E_e) &= -\frac{1}{M} \frac{1}{2\lambda(1+\lambda)(1+3\lambda^2)} \left(A_1^{(W)} E_0 + A_2^{(W)} E_e + A_3^{(W)} \frac{m_e^2}{E_e} \right) = \\ &= -8.55 \times 10^{-3} + 3.73 \times 10^{-2} \frac{E_e}{E_0} + 1.42 \times 10^{-3} \frac{m_e}{E_e}. \end{aligned} \quad (21)$$

In the experimentally used region of electron kinetic energies $250 \text{ keV} \leq T_e \leq 455 \text{ keV}$ [1, 37, 39] the radiative corrections $(\alpha/\pi) f_n(E_e)$ vary over the region $1.53 \times 10^{-3} \geq (\alpha/\pi) f_n(E_e) \geq 1.04 \times 10^{-3}$ and increase the absolute value of the correlation coefficient $A^{(W)}(E_e)$.

Summary. The Wilkinson expression for the correlation coefficient $A^{(W)}(E_e)$ (see Eq.(20)), taking into account a complete set of the $1/M$ corrections from the “weak magnetism” and the proton recoil, is improved by the account for

the radiative corrections, described by the function $2.81 \times 10^{-3} \geq (\alpha/\pi) f_n(E_e) \geq 0.62 \times 10^{-3}$ for the electron energies $m_e \leq E_e \leq E_0$ (see Eq.(D-58)). The contribution of the radiative corrections increases the absolute value of the correlation coefficient $A^{(W)}(E_e)$. The results, obtained in this section, should be used as a theoretical background for the experimental determination of contributions of order 10^{-4} of interactions beyond the SM from the experimental data on the electron asymmetry $A_{\text{exp}}(E_e)$ (see section IX).

V. STANDARD MODEL ANALYSIS OF EXPERIMENTAL DETERMINATION OF CORRELATION COEFFICIENT B_0 . ANTINEUTRINO ASYMMETRY $B_{\text{exp}}(E_e)$

In the SM to leading order in the large M expansion the correlations between the neutron spin and the antineutrino 3-momentum are defined by the correlation coefficient B_0 [1], which may be determined from the experimental data on the asymmetry $B_{\text{exp}}(E_e)$, defined by [38]

$$B_{\text{exp}}(E_e) = \frac{N^{--}(E_e) - N^{++}(E_e)}{N^{--}(E_e) + N^{++}(E_e)}. \quad (22)$$

It defines the asymmetry of the emission of the antineutrinos into the forward and backward hemisphere with respect to the neutron spin, where $N^{\mp\mp}(E_e)$ is the number of events of the emission of the electron-proton pairs as functions of the electron energy E_e . The signs $(++)$ and $(--)$ show that the electron-proton pairs were emitted in $(++)$ and against $(--)$ a direction of the neutron spin. This means that antineutrinos were emitted against $(++)$ and in $(--)$ a direction of the neutron spin. The number of events $N^{--}(E_e)$ and $N^{++}(E_e)$ are defined by the electron-energy and angular distribution of the neutron β^- -decay, integrated over the forward and backward hemisphere relative to the neutron spin, respectively.

The integration region for the electron-proton pairs, emitted in the direction $(++)$ of the neutron spin, is defined by the constraints [64]: $\vec{\xi}_n \cdot \vec{k}_e = P k_e \cos \theta_e > 0$ and $\vec{\xi}_n \cdot \vec{k}_p = \vec{\xi}_n \cdot (-\vec{k}_e - \vec{k}) = P E(-r \cos \theta_e - \cos \theta) > 0$ or $-r \cos \theta_e > \cos \theta$, where $r = k_e/E = \sqrt{E_e^2 - m_e^2}/(E_0 - E_e)$ and $P = |\vec{\xi}_n| \leq 1$ is the neutron polarisation. For $N^{++}(E_e)$ and $r < 1$ we obtain the following expression

$$N^{++}(E_e) = 2\pi N(E_e) \left\{ \left(1 - \frac{1}{4} a\beta + \frac{1}{2} P\beta \left(A - \frac{1}{3} K_n \beta \right) - \frac{1}{2} P \left(B - \frac{1}{3} Q_n \beta \right) \right) - \frac{1}{2} r \left(1 + \frac{2}{3} P A \beta \right) + \frac{1}{8} r^2 \left(a\beta + \frac{4}{3} P B + \frac{4}{5} P K_n \beta^2 \right) - \frac{1}{15} r^3 P Q_n \beta - \frac{1}{8} a_0 \beta^2 r (1 - r^2) \frac{E_e}{M} \right\}, \quad (23)$$

where $N(E_e)$ is the normalisation factor Eq.(19). For $r > 1$ the upper limit of the integration over $\cos \theta_e$ is restricted by $\cos \theta_e \leq 1/r$. The result of the interaction is

$$N^{++}(E_e) = 2\pi N(E_e) \left\{ \frac{1}{2} \frac{1}{r} \left(1 - \frac{2}{3} P B \right) - \frac{1}{8} \frac{1}{r^2} \left(a\beta - \frac{4}{3} P A \beta - \frac{4}{5} P Q_n \beta \right) - \frac{1}{15} \frac{1}{r^3} P K_n \beta^2 + \frac{1}{8} a_0 \beta^2 \frac{1}{r} \left(1 - \frac{1}{r^2} \right) \frac{E_e}{M} \right\}. \quad (24)$$

For the calculation of $N^{--}(E_e)$ we have to integrate over the region [64]: $\vec{\xi}_n \cdot \vec{k}_e = P k_e \cos \theta_e < 0$ and $\vec{\xi}_n \cdot \vec{k}_p = P E(-r \cos \theta_e - \cos \theta) < 0$ or $\cos \theta > -r \cos \theta_e$. The number of events $N^{--}(E_e)$ for $r < 1$ is given by

$$N^{--}(E_e) = 2\pi N(E_e) \left\{ \left(1 - \frac{1}{4} a\beta - \frac{1}{2} P\beta \left(A - \frac{1}{3} K_n \beta \right) + \frac{1}{2} P \left(B - \frac{1}{3} Q_n \beta \right) \right) - \frac{1}{2} r \left(1 - \frac{2}{3} P A \beta \right) + \frac{1}{8} r^2 \left(a\beta - \frac{4}{3} P B - \frac{4}{5} P K_n \beta^2 \right) + \frac{1}{15} r^3 P Q_n \beta - \frac{1}{8} a_0 \beta^2 r (1 - r^2) \frac{E_e}{M} \right\}. \quad (25)$$

For $r > 1$ the lower limit of the integration over $\cos \theta_e$ is restricted by $\cos \theta_e > -1/r$. The number of events $N^{--}(E_e)$, calculated for $r > 1$, is equal to

$$N^{--}(E_e) = 2\pi N(E_e) \left\{ \frac{1}{2} \frac{1}{r} \left(1 + \frac{2}{3} P B \right) - \frac{1}{8} \frac{1}{r^2} \left(a\beta + \frac{4}{3} P A \beta + \frac{4}{5} P Q_n \beta \right) + \frac{1}{15} \frac{1}{r^3} P K_n \beta^2 + \frac{1}{8} a_0 \beta^2 \frac{1}{r} \left(1 - \frac{1}{r^2} \right) \frac{E_e}{M} \right\}. \quad (26)$$

Using our formulas for the numbers of events we calculate the asymmetry $B_{\text{exp}}(E_e)$. For $r \leq 1$ or $0 \leq T_e \leq (E_0 - m_e)^2/2E_0 = 236 \text{ keV}$ and for $r \geq 1$ or $(E_0 - m_e)^2/2E_0 = 236 \text{ keV} \leq T_e \leq E_0 - m_e$ the asymmetry $B_{\text{exp}}(E_e)$ is

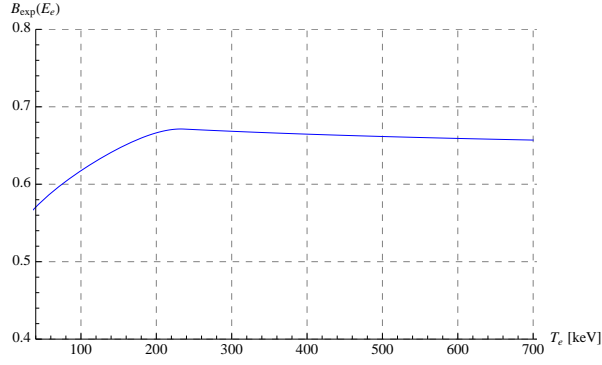


FIG. 2: The antineutrino asymmetry $B_{\text{exp}}(E_e)$, including a complete set of the $1/M$ corrections, caused by the “weak magnetism” and the proton recoil, and radiative corrections of order α/π , calculated to leading order in the large M expansion; $T_e = E_e - m_e$ is the electron kinetic energy.

equal to

$$B_{\text{exp}}(E_e) = \frac{4P}{3} \left(B(3 - r^2) - (3 - 2r)A\beta + \left(1 - \frac{3}{5}r^2\right) K_n \beta^2 - \left(1 - \frac{2}{5}r^3\right) Q_n \beta \right) \times \left((8 - 4r) - (2 - r^2)a\beta - a_0 \beta^2 r(1 - r^2) \frac{E_e}{M} \right)^{-1}. \quad (27)$$

and

$$B_{\text{exp}}(E_e) = \frac{2P}{3} \left(B - \frac{1}{2} \left(A + \frac{3}{5}Q_n \right) \frac{\beta}{r} + \frac{1}{5}K_n \frac{\beta^2}{r^2} \right) \left(1 - a \frac{\beta}{4r} + \frac{1}{4} a_0 \beta^2 \left(1 - \frac{1}{r^2} \right) \frac{E_e}{M} \right)^{-1}, \quad (28)$$

respectively. At $r = 1$ or $T_e = (E_0 - m_e)^2/2E_0 = 236 \text{ keV}$ the asymmetry $B_{\text{exp}}(E_e)$ is continuous. To leading order in the large M expansion the asymmetry $B_{\text{exp}}(E_e)$ reduces to the form

$$B_{\text{exp}}(E_e) = \frac{2P}{3} \begin{cases} \frac{B\left(\frac{3}{2} - \frac{1}{2}r^2\right) - \left(\frac{3}{2} - r\right)A\beta}{(2 - r) - \frac{1}{2}\left(1 - \frac{1}{2}r^2\right)a\beta}, & r \leq 1 \\ \frac{B - \frac{1}{2}A\frac{\beta}{r}}{1 - \frac{1}{4}a\frac{\beta}{r}}, & r \geq 1, \end{cases} \quad (29)$$

where the correlation coefficients B , A and a are equal to

$$B = B_0, \quad A = A_0 \left(1 + \frac{\alpha}{\pi} f_n(E_e) \right), \quad a = a_0 \left(1 + \frac{\alpha}{\pi} f_n(E_e) \right). \quad (30)$$

In Fig. 2 we plot the asymmetry $B_{\text{exp}}(E_e)$, given by Eq.(27) and Eq.(28) and obtained in the SM with the account for the contributions of the $1/M$ corrections, caused by the “weak magnetism” and the proton recoil and the radiative corrections of order α/π , calculated to leading order in the large M expansion and described by the function $(\alpha/\pi) f_n(E_e)$. In [38, 39] for the experimental determination of the correlation coefficient B_0 the experimental data on the asymmetry $B_{\text{exp}}(E_e)$ were fitted by Eq.(29) with the replacement $B \rightarrow B_0$, $A \rightarrow A_0$ and $a \rightarrow a_0$, respectively. Such an asymmetry was calculated in [64].

The theoretical value $B_0 = 0.9871(1)$ of the correlation coefficient B_0 , calculated for $\lambda = -1.2750(9)$, agrees 1) within 1.5 standard deviations with the experimental values $B_0^{(\text{exp})} = 0.9802(50)$, $B_0^{(\text{exp})} = 0.9801(46)$ and $B_0^{(\text{exp})} = 0.9894(83)$, obtained in [38, 39] (see also [1]), [65] and [66], respectively, and within two standard deviations with the experimental one $B_0^{(\text{exp})} = 0.967(12)$, obtained in [67].

Summary. The antineutrino asymmetry $B_{\text{exp}}(E_e)$, calculated in [64] and used in [38, 39] for the experimental determination of the correlation coefficient $B_0^{(\text{exp})} = 0.9802(50)$, is improved by the contributions of a complete set of the $1/M$ corrections, caused by the “weak magnetism” and the proton recoil, and the radiative corrections $(\alpha/\pi) f_n(E_e)$, calculated to leading order in the large M expansion. The obtained expression of the antineutrino asymmetry $B_{\text{exp}}(E_e)$ should be used as a theoretical background for the experimental determination of contributions of order 10^{-4} of interactions beyond the SM (see section IX).

VI. STANDARD MODEL ANALYSIS OF EXPERIMENTAL DETERMINATION OF CORRELATION COEFFICIENT a_0 . ELECTRON-PROTON ENERGY DISTRIBUTION $a(E_e, T_p)$ AND PROTON-ENERGY SPECTRUM $a(T_p)$

For the experimental determination of the correlation coefficient a_0 the correlation coefficient $a(E_e)$, given by Eq.(11), can be hardly used, since the antineutrino is hard to detect. Thus, for the determination of a_0 and the contributions of interactions beyond the SM one should measure correlations of the 3-momenta of the decay charged particles. Using the results, obtained in Appendix I, the electron-proton energy spectrum for the neutron β^- -decay with unpolarised particles can be given in the following form

$$\frac{d^2\text{BR}_{\beta_c^-}(E_e, T_p)}{dE_e dT_p} = (\tau_n)_{\text{SM}} M (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{4\pi^3} a(E_e, T_p) \left(1 + \frac{\alpha}{\pi} g_n(E_e)\right) F(E_e, Z=1) E_e, \quad (31)$$

where $(\tau_n)_{\text{SM}}$ is the theoretical lifetime of the neutron, calculated in the SM (see section VIII) and T_p is the kinetic energy of the proton varying from zero to its maximal value $(T_p)_{\text{max}} = (m_n - m_p)^2 - m_e^2 / 2m_n = (E_0^2 - m_e^2) / 2M = 0.751 \text{ keV}$, i.e. $0 \leq T_p \leq 0.751 \text{ keV}$. The limits of the integration over the electron energy E_e , i.e. $(E_e)_{\text{min}} \leq E_e \leq (E_e)_{\text{max}}$, are the functions of the proton kinetic energy T_p . They are adduced in Appendix I. The electron-proton energy distribution $a(E_e, T_p)$ is defined by (see Appendix I)

$$a(E_e, T_p) = \zeta_1(E_e, T_p) + a_0 \left(1 + \frac{1}{1 - \lambda^2} \frac{E_0}{M}\right) \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) \zeta_2(E_e, T_p). \quad (32)$$

The functions $\zeta_1(E_e, T_p)$ and $\zeta_2(E_e, T_p)$ are given by in Appendix I. They are calculated to leading order in the large M and do not contain the radiative corrections. In the electron energy region $(E_e)_{\text{min}} \leq E_e \leq (E_e)_{\text{max}}$ the contribution of the radiative corrections $(\alpha/\pi) f_n(E_e)$ to the electron-proton energy distribution $a(E_e, T_p)$ relative to the correlation coefficient a_0 is of order 10^{-3} . The account for the radiative corrections is very important for a correct experimental determination of contributions of order 10^{-4} of interactions beyond the SM.

Integrating the electron-proton energy spectrum Eq.(31) over the electron energy $(E_e)_{\text{min}} \leq E_e \leq (E_e)_{\text{max}}$ we obtain the proton-energy spectrum

$$\frac{d\text{BR}_{\beta_c^-}(T_p)}{dT_p} = (\tau_n)_{\text{SM}} M (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{4\pi^3} a(T_p), \quad (33)$$

where $a(T_p)$ is defined by

$$a(T_p) = g_1(T_p) + a_0 \left(1 + \frac{1}{1 - \lambda^2} \frac{E_0}{M}\right) g_2(T_p). \quad (34)$$

The functions $g_1(T_p)$ and $g_2(T_p)$ are given in Appendix I. Recently the correlation coefficient a_0 and the axial coupling constant λ have been determined from the proton-energy spectrum by Byrne *et al.* [40]. The obtained value $a_0^{(\text{exp})} = -0.1054(55)$ can be fitted by the axial coupling constant $\lambda = -1.271(18)$. In turn, the experimental value $a_0^{(\text{exp})} = -0.1017(51)$, obtained in [41], defines the axial coupling constant equal to $\lambda = -1.259(17)$. These experimental values agree with the theoretical value of the correlation coefficient $a_0 = -0.1065(3)$, calculated at $\lambda = -1.2750(9)$, and with the axial coupling constant $\lambda = -1.2750(9)$ within one standard deviation.

From the point of view of the experimental determination of contributions of order 10^{-4} of interactions beyond the SM (see a discussion in section VII) the experimental accuracy of the determination of the axial coupling constant λ by measuring the proton-energy spectrum as well as the electron-proton energy distribution $a(E_e, T_p)$ should be improved by more than two orders of magnitude in comparison with the accuracy of Byrne's experiment [40]. There are two major, funded experiments, which are currently attempting to do this. They are 1) aCORN at the National Institute of Standards and Technology (NIST) and 2) Nab at the new Spallation Neutron Source (SNS) in Oak Ridge at Tennessee [68]. The expected experimental accuracy of the determination of the correlation coefficient a_0 in the aCORN and Nab experiments is better than 1 % (see also [69]).

We have discussed the experimental determination of the correlation coefficient a_0 by measuring the electron-proton energy distribution $a(E_e, T_p)$ and the proton-energy spectrum $a(T_p)$. However, there are another possibilities to determine experimentally the correlation coefficient a_0 . It can be extracted from the experimental data on the $(T_e, \cos \theta_{e\bar{\nu}})$ and $(T_e, \cos \theta_{ep})$ distributions [70], where $\theta_{e\bar{\nu}}$ and θ_{ep} are angles of the electron-antineutrino and electron-proton correlations, respectively.

Summary. The electron-proton energy distribution $a(E_e, T_p)$ is calculated by taking into account a complete set of the $1/M$ corrections, caused by the “weak magnetism” and the proton recoil, and the radiative corrections of

order α/π , calculated to leading order in the large M expansion. The proton–energy spectrum is improved in comparison with that, used in [40], by the account for the $1/M$ and radiative corrections. The obtained expression for the electron–proton energy distribution $a(E_e, T_p)$ as well as the proton–energy spectrum $a(T_p)$ should provide a theoretical background for the experimental determination of contributions of order 10^{-4} of interactions beyond the SM (see section IX). For the analysis of contributions of interactions beyond the SM we propose to multiply the electron–proton energy spectrum of the rate of the neutron β^- –decay by the lifetime of the neutron τ_n , calculated with the account for the contributions of interactions beyond the SM (see Appendix G). As a result the analysis of the contributions of vector and axial–vector interactions beyond the SM by means of the electron–proton energy distribution $\tau_n a(E_e, T_p)$ and the proton–energy spectrum $\tau_n a(T_p)$ reduces to the analysis of these contributions to the axial coupling constant λ only (see section IX).

VII. STANDARD MODEL ANALYSIS OF EXPERIMENTAL DETERMINATION OF CORRELATION COEFFICIENT C_0 . PROTON ASYMMETRY C_{exp}

The correlations between the neutron spin and the proton 3–momentum are described by the correlation coefficient C [71]–[75]. It defines the proton recoil asymmetry $C_0 = -x_C(A_0 + B_0)$, where x_C is the theoretical numerical factor, calculated within the SM to leading order in the large M expansion. The first measurement of the correlation coefficient $C_0^{(\text{exp})} = -0.2377(26)$ has been performed by Schumann *et al.* [42]. The angular distribution of the probability of the neutron β^- –decay, related to the proton recoil asymmetry, is given by [71]

$$4\pi \frac{dW(\theta_p)}{d\Omega_p} = 1 + 2PC \cos \theta_p, \quad (35)$$

where $d\Omega_p = 2\pi \sin \theta_p d\theta_p$ is the infinitesimal solid angle of the proton 3–momentum with respect to the neutron polarisation $\vec{\xi}_n$, i.e. $\vec{\xi}_n \cdot \vec{k}_p = P k_p \cos \theta_p$ and $P = |\vec{\xi}_n|$ is the neutron polarisation. The correlation coefficient C , calculated at the account for the $1/M$ and radiative of order α/π corrections, is equal to (see Appendix I Eq.(I-29))

$$\begin{aligned} C = & -\frac{1}{2} \frac{X_8}{X_1} (A_0 + B_0) + \frac{1}{2} \frac{X_9}{X_1} A_0 + \frac{\alpha}{\pi} \frac{1}{2} \frac{X_{10}}{X_1} A_0 - \frac{\alpha}{\pi} \frac{1}{2} \frac{X_{11}}{X_1} B_0 + \frac{1}{M} \frac{1}{1+3\lambda^2} \left(\lambda \frac{1}{2} \frac{X_{12}}{X_1} - (\kappa+1) \lambda \frac{1}{2} \frac{X_{13}}{X_1} \right. \\ & - (2\kappa+1) \lambda \frac{1}{2} \frac{X_{14}}{X_1} - \lambda(1+\lambda) \frac{1}{2} \frac{X_{15}+Y_3}{X_1} + \lambda(1-\lambda) \frac{1}{2} \frac{X_{16}+Y_4}{X_1} \Big) - \frac{\alpha}{\pi} \frac{1}{2} \frac{X_2}{X_1} - \frac{1}{M} \frac{1}{1+3\lambda^2} \left(\frac{1}{2} \frac{X_3}{X_1} + (1+3\lambda^2) \right. \\ & \times \frac{1}{2} \frac{X_4+Y_1}{X_1} - (1-\lambda^2) \frac{1}{2} \frac{X_5+Y_2}{X_1} + (\lambda^2+2(\kappa+1)\lambda) \frac{1}{2} \frac{X_6}{X_1} - (\lambda^2-2(\kappa+1)\lambda) \frac{1}{2} \frac{X_7}{X_1} \Big), \end{aligned} \quad (36)$$

where the numerical factors X_j ($j = 1, \dots, 14$) and Y_j ($j = 1, 2, 3, 4$) are calculated in Appendix I (see Eq.(I-22) and Eq.(I-26)). The factor $x_C = X_8/2X_1 = 0.27491$ agrees well with the factor $x_C = 0.27594$, calculated by Glück [72]. As a result the correlation coefficient C_0 is equal to

$$C_0 = -0.27491 (A_0 + B_0). \quad (37)$$

The numerical value $C_0 = -0.2386$, calculated at $\lambda = -1.2750$, agrees well with the experimental one $C_0^{\text{exp}} = -0.2377(26)$ [42]. Defining the proton recoil asymmetry C_{exp} as we have defined the asymmetry $A_{\text{exp}}(E_e)$ (see Eq.(17)) we obtain

$$C_{\text{exp}} = PC (\cos \vartheta_1 + \cos \vartheta_2), \quad (38)$$

where the polar angles θ_1 and θ_2 define the solid angle $\Delta\Omega_{12} = 2\pi(\cos \theta_1 - \cos \theta_2)$ of the proton emission to the forward and backward hemisphere with respect to the neutron spin.

Summary. The correlation coefficient C , describing correlations between the neutron spin and the proton 3–momentum, is calculated by taking into account a complete set of the $1/M$ corrections, caused by the “weak magnetism” and the proton recoil, and the radiative corrections of order α/π , calculated to leading order in the large M expansion. The obtained result should be used as a theoretical background for the experimental determination of contributions of order 10^{-4} of interactions beyond the SM (see section IX).

VIII. STANDARD MODEL ANALYSIS OF LIFETIME OF NEUTRON

Having integrated the electron–energy and angular distribution Eq.(6) over the directions of the electron 3–momentum \vec{k}_e , the antineutrino 3–momentum \vec{k} and the electron energy E_e within the limits $m_e \leq E_e \leq E_0$,

we obtain the rate of the neutron β^- -decay. It is equal to [10]

$$(\lambda_n)_{\text{SM}} = (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{2\pi^3} f_n(E_0, Z = 1), \quad (39)$$

where the Fermi integral $f_n(E_0, Z = 1)$, given by

$$\begin{aligned} f_n(E_0, Z = 1) &= \int_{m_e}^{E_0} (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) \left(1 + \frac{\alpha}{\pi} g_n(E_e)\right) \\ &\times \left\{1 + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left[\left(10\lambda^2 - 4(\kappa + 1)\lambda + 2\right) E_e - 2\lambda \left(\lambda - (\kappa + 1)\right) \left(E_0 + \frac{m_e^2}{E_e}\right) \right] \right\} dE_e, \end{aligned} \quad (40)$$

contains the contributions of the “weak magnetism”, the proton recoil and the radiative corrections, described by the function $g_n(E_e)$. The calculation of the lifetime of the neutron with the Fermi function, determined by Eq.(5), the axial coupling constant $\lambda = -1.2750(9)$ and the CKM matrix element $V_{ud} = 0.97428(15)$, gives $(\tau_n)_{\text{SM}} = 879.6(1.1)$ s. The error bars ± 1.1 s are defined by the error bars of the experimental value of the coupling constant and the CKM matrix element. The theoretical value of the lifetime of the neutron $(\tau_n)_{\text{SM}} = 879.6(1.1)$ s agrees well with the experimental values $\tau_n^{(\text{exp})} = 878.5(8)$ s and $\tau_n^{(\text{exp})} = 880.7(1.8)$ s and $\tau_n^{(\text{exp})} = 881.6(2.1)$ s, measured by Serebrov *et al.* [12], Pilchmaier *et al.* [76] and Arzumanov *et al.* [77], respectively. It agrees also well with the new world average values (w.a.v.) of the neutron lifetime $\tau_n^{(\text{w.a.v.})} = 880.1(1.1)$ s [9], $\tau_n^{(\text{w.a.v.})} = 880.0(9)$ s [78] and $\tau_n^{(\text{w.a.v.})} = 881.9(1.3)$ s [79], respectively.

Summary. The lifetime of the neutron is calculated by taking into account the radiative corrections by Sirlin *et al.*, calculated to leading order in the large M expansion, and a complete set of the $1/M$ corrections, caused by the “weak magnetism” and the proton recoil. The obtained result should be used as a theoretical background for the experimental determination of contributions of order 10^{-4} of interactions beyond the SM (see section IX). The numerical value of the lifetime of the neutron $(\tau_n)_{\text{SM}} = 879.6(1.1)$ s, calculated in this section, is left unchanged after the absorption of the Herczeg phenomenological coupling constants a_{LL}^h and a_{LR}^h of the left–left and left–right lepton–nucleon current–current interactions (vector and axial–vector interactions beyond the SM) by the axial coupling constant $\lambda \rightarrow \lambda_{\text{eff}} = (\lambda - a_{LL}^h + a_{LR}^h)/(1 + a_{LL}^h + a_{LR}^h) = \lambda - a_{LL}^h + a_{LR}^h - \lambda(a_{LL}^h + a_{LR}^h)$ and the CKM matrix element $V_{ud} \rightarrow (V_{ud})_{\text{eff}} = V_{ud}(1 + a_{LL}^h + a_{LR}^h)$ (see section IX).

IX. SENSITIVITY OF ELECTRON–PROTON ENERGY DISTRIBUTION, ASYMMETRIES $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$ AND C_{exp} AND LIFETIME OF NEUTRON τ_n TO CONTRIBUTIONS OF ORDER 10^{-4} OF INTERACTIONS BEYOND STANDARD MODEL

In this section we propose a theoretical analysis of the sensitivity of the electron–proton energy distribution $a(E_e, T_p)$, the proton–energy spectrum $a(T_p)$, the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$ and C_{exp} and the lifetime of the neutron τ_n to contributions of order 10^{-4} of interactions beyond the SM.

The electron–proton energy distribution $a(E_e, T_p)$ and the correlation coefficients $a(E_e)$, $A(E_e)$ and $B(E_e)$, including the contributions of interactions beyond the SM, can be written in the form

$$\begin{aligned} a(E_e, T_p) &= \zeta_1(E_e, T_p)_{\text{eff}} + \bar{a}_{\text{eff}}(E_e) \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) \zeta_2(E_e, T_p), \\ a(E_e) &= a_{\text{eff}}(E_e) \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right), \\ A(E_e) &= A_{\text{eff}}(E_e) \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right), \\ B(E_e) &= B_{\text{eff}}(E_e), \end{aligned} \quad (41)$$

where $\zeta_1(E_e, T_p)_{\text{eff}}$, $\bar{a}_{\text{eff}}(E_e)$, $a_{\text{eff}}(E_e)$, $A_{\text{eff}}(E_e)$ and $B_{\text{eff}}(E_e)$ are the electron–proton energy distribution and the correlation coefficients, defined by the contributions of the “weak magnetism”, the proton recoil and interactions

beyond the SM only. They are given by

$$\begin{aligned}
\zeta_1(E_e, T_p)_{\text{eff}} &= \left(1 - \frac{a_4}{a_0} \left(\frac{m_e}{E_e} - \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right) \right) \zeta_1(E_e, T_p), \\
\bar{a}_{\text{eff}}(E_e) &= (a_0)_{\text{eff}} + \frac{1}{M} \frac{1}{1+3\lambda^2} E_0 + a_4 \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}}, \\
a_{\text{eff}}(E_e) &= (a_0)_{\text{eff}} + \frac{1}{M} \frac{1}{(1+3\lambda^2)^2} \left(a_1 E_0 + a_2 E_e + a_3 \frac{m_e^2}{E_e} \right) + a_4 \frac{m_e}{E_e}, \\
A_{\text{eff}}(E_e) &= (A_0)_{\text{eff}} + \frac{1}{M} \frac{1}{(1+3\lambda^2)^2} \left(A_1 E_0 + A_2 E_e + A_3 \frac{m_e^2}{E_e} \right) + A_4 \frac{m_e}{E_e}, \\
B_{\text{eff}}(E_e) &= (B_0)_{\text{eff}} + \frac{1}{M} \frac{1}{(1+3\lambda^2)^2} \left(B_1 E_0 + B_2 E_e + B_3 \frac{m_e^2}{E_e} \right) + B_4 \frac{m_e}{E_e},
\end{aligned} \tag{42}$$

where $\langle m_e/E_e \rangle_{\text{SM}}$ is the average value, calculated in the SM with the electron–energy spectrum density Eq.(D-59). The terms, proportional to $\langle m_e/E_e \rangle_{\text{SM}}$, come from the lifetime of the neutron, calculated at the account for the contributions of interactions beyond the SM (see Appendix G). The coefficients a_j , A_j and B_j for $j = 1, 2, 3$ are given in Eq.(11), Eq.(12) and Eq.(13), respectively. They are calculated to next-to-leading order in the large M expansion and include the contributions of the “weak magnetism” and the proton recoil only. The numerical values of these corrections are estimated in Eq.(14) at $\lambda = -1.2750$. The terms $(a_0)_{\text{eff}}$, $(A_0)_{\text{eff}}$ and $(B_0)_{\text{eff}}$ are the sums of the correlation coefficients a_0 , A_0 and B_0 and energy independent contributions of interactions beyond the SM (see Appendix G). They read

$$\begin{aligned}
(a_0)_{\text{eff}} &= a_0 + \frac{1}{(1+3\lambda^2)^2} \left(4\lambda^2 \text{Re}(\delta C_V - \delta \bar{C}_V) + 4\lambda \text{Re}(\delta C_A - \delta \bar{C}_A) \right), \\
(A_0)_{\text{eff}} &= A_0 + \frac{1}{(1+3\lambda^2)^2} \left(-\lambda(3\lambda^2 - 2\lambda - 1) \text{Re}(\delta C_V - \delta \bar{C}_V) - (3\lambda^2 - 2\lambda - 1) \text{Re}(\delta C_A - \delta \bar{C}_A) \right), \\
(B_0)_{\text{eff}} &= B_0 + \frac{1}{(1+3\lambda^2)^2} \left(-\lambda(3\lambda^2 + 2\lambda - 1) \text{Re}(\delta C_V - \delta \bar{C}_V) - (3\lambda^2 + 2\lambda - 1) \text{Re}(\delta C_A - \delta \bar{C}_A) \right).
\end{aligned} \tag{43}$$

The coefficients a_4 , A_4 and B_4 are induced by interactions beyond the SM only. They are equal to

$$\begin{aligned}
a_4 &= -\frac{1-\lambda^2}{(1+3\lambda^2)^2} \left(\text{Re}(C_S - \bar{C}_S) + 3\lambda \text{Re}(C_T - \bar{C}_T) \right), \\
A_4 &= 2 \frac{\lambda(1+\lambda)}{(1+3\lambda^2)^2} \left(\text{Re}(C_S - \bar{C}_S) + 3\lambda \text{Re}(C_T - \bar{C}_T) \right), \\
B_4 &= \frac{(1+\lambda)(1-3\lambda)}{(1+3\lambda^2)^2} \left(\lambda \text{Re}(C_S - \bar{C}_S) - \text{Re}(C_T - \bar{C}_T) \right).
\end{aligned} \tag{44}$$

For the derivation of Eqs.(43) and (44) we have used Eq.(G-8) in Appendix G with the coupling constants $C_V = 1 + \delta C_V$, $\bar{C}_V = -1 + \delta \bar{C}_V$, $C_A = -\lambda + \delta C_A$ and $\bar{C}_A = \lambda + \delta \bar{C}_A$ and have kept only the linear contributions of the deviations from the coupling constants of the SM. By assumption the axial coupling constant λ in Eqs.(43) and (44) is determined by all interactions within the SM only.

The differences of the phenomenological coupling constants $\delta C_V - \delta \bar{C}_V$, $\delta C_A - \delta \bar{C}_A$, $C_S - \bar{C}_S$ and $C_T - \bar{C}_T$, calculated in terms of the Herczeg phenomenological lepton–nucleon coupling constants [6] (see Appendix G), take the form

$$\begin{aligned}
\delta C_V - \delta \bar{C}_V &= 2(a_{LL}^h + a_{LR}^h), \\
\delta C_A - \delta \bar{C}_A &= 2(a_{LL}^h - a_{LR}^h), \\
C_S - \bar{C}_S &= 2(A_{LL}^h + A_{LR}^h) = 4a_S, \\
C_T - \bar{C}_T &= 4\alpha_{LL}^h = 4a_T.
\end{aligned} \tag{45}$$

This means that the phenomenological coupling constants $\delta C_V - \delta \bar{C}_V$, $\delta C_A - \delta \bar{C}_A$, $C_S - \bar{C}_S$ and $C_T - \bar{C}_T$ describe interactions beyond the SM of left–handed leptonic currents and left(right)–handed hadronic currents, i.e. $L \otimes L$ and $L \otimes R$, respectively.

The experimental determination of the correlation coefficients a_0 , A_0 and B_0 runs as follows. Using the theoretical expressions for the electron–proton energy distribution $a(E_e, T_p)$ and the asymmetries $A_{\text{exp}}(E_e)$ and $B_{\text{exp}}(E_e)$ the experimental data on them are being fitted by a tuning of the axial coupling constant [1]. Such a procedure gives the

axial-coupling constants λ_a , λ_A and λ_B , obtained from the fit of the experimental data on $a(E_e, T_p)$, $A_{\text{exp}}(E_e)$ and $B_{\text{exp}}(E_e)$, respectively. In terms of these axial coupling constants one may define the correlation coefficients $(a_0)_{\text{eff}}^{(\text{exp})}$, $(A_0)_{\text{eff}}^{(\text{exp})}$ and $(B_0)_{\text{eff}}^{(\text{exp})}$, respectively, as follows

$$(a_0)_{\text{eff}}^{(\text{exp})} = \frac{1 - \lambda_a^2}{1 + 3\lambda_a^2} \quad , \quad (A_0)_{\text{eff}}^{(\text{exp})} = -2 \frac{\lambda_A(1 + \lambda_A)}{1 + 3\lambda_A^2} \quad , \quad (B_0)_{\text{eff}}^{(\text{exp})} = -2 \frac{\lambda_B(1 - \lambda_B)}{1 + 3\lambda_B^2} \quad (46)$$

The theoretical expressions for the axial coupling constants λ_a , λ_A and λ_B in terms of the axial coupling constant λ , defined by interactions within the SM only, and the contributions of interactions beyond the SM are

$$\begin{aligned} \lambda_a &= \lambda - \frac{1}{2} \left(\lambda \text{Re}(\delta C_V - \delta \bar{C}_V) + \text{Re}(\delta C_A - \delta \bar{C}_A) \right), \\ \lambda_A &= \lambda - \frac{1}{2} \left(\lambda \text{Re}(\delta C_V - \delta \bar{C}_V) + \text{Re}(\delta C_A - \delta \bar{C}_A) \right), \\ \lambda_B &= \lambda - \frac{1}{2} \left(\lambda \text{Re}(\delta C_V - \delta \bar{C}_V) + \text{Re}(\delta C_A - \delta \bar{C}_A) \right). \end{aligned} \quad (47)$$

Thus, in the linear approximation with respect to the deviations of the phenomenological coupling constants of interactions beyond the SM from the coupling constants of the SM we obtain that $\lambda_a = \lambda_A = \lambda_B$. This agrees well with the results obtained in [43]–[45]. Replacing λ_a , λ_A and λ_B by λ_{eff} , which includes the contributions of vector and axial-vector interactions beyond the SM in addition to the contributions of interactions within the SM, and denoting

$$b_F = \frac{1}{1 + 3\lambda_{\text{eff}}^2} \left(\text{Re}(C_S - \bar{C}_S) + 3\lambda_{\text{eff}} \text{Re}(C_T - \bar{C}_T) \right) = \frac{4}{1 + 3\lambda_{\text{eff}}^2} \left(\text{Re}(a_S) + 3\lambda_{\text{eff}} \text{Re}(a_T) \right) \quad (48)$$

and

$$c_{ST} = \frac{1}{1 + 3\lambda_{\text{eff}}^2} \left(\lambda_{\text{eff}} \text{Re}(C_S - \bar{C}_S) - \text{Re}(C_T - \bar{C}_T) \right) = \frac{4}{1 + 3\lambda_{\text{eff}}^2} \left(\lambda_{\text{eff}} \text{Re}(a_S) - \text{Re}(a_T) \right), \quad (49)$$

where b_F is the Fierz term (see Appendix G), we obtain the electron–proton energy distribution $a(E_e, T_p)$ and the correlation coefficients $a(E_e)$, $A(E_e)$ and $B(E_e)$ in the following form

$$\begin{aligned} a(E_e, T_p) &= \left(1 - b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right) \left\{ \left(1 + b_F \frac{m_e}{E_e} \right) \zeta_1(E_e, T_p) + a_0 \left(1 + \frac{1}{1 - \lambda_{\text{eff}}^2} \frac{E_0}{M} \right) \left(1 + \frac{\alpha}{\pi} f_n(E_e) \right) \zeta_2(E_e, T_p) \right\}, \\ a(E_e) &= a_0 \left(1 - b_F \frac{m_e}{E_e} \right) \left(1 + \frac{\alpha}{\pi} f_n(E_e) \right) \left\{ 1 + \frac{1}{M} \frac{1}{(1 - \lambda_{\text{eff}}^2)(1 + 3\lambda_{\text{eff}}^2)} \left(a_1 E_0 + a_2 E_e + a_3 \frac{m_e^2}{E_e} \right) \right\}, \\ A(E_e) &= A_0 \left(1 - b_F \frac{m_e}{E_e} \right) \left(1 + \frac{\alpha}{\pi} f_n(E_e) \right) \left\{ 1 - \frac{1}{M} \frac{1}{2\lambda_{\text{eff}}(1 + \lambda_{\text{eff}})(1 + 3\lambda_{\text{eff}}^2)} \left(A_1 E_0 + A_2 E_e + A_3 \frac{m_e^2}{E_e} \right) \right\}, \\ B(E_e) &= B_0 \left(1 - \frac{(1 + \lambda_{\text{eff}})(1 - 3\lambda_{\text{eff}})}{2\lambda_{\text{eff}}(1 - \lambda_{\text{eff}})} c_{ST} \frac{m_e}{E_e} \right) \left\{ 1 - \frac{1}{M} \frac{1}{2\lambda_{\text{eff}}(1 - \lambda_{\text{eff}})(1 + 3\lambda_{\text{eff}}^2)} \left(B_1 E_0 + B_2 E_e + B_3 \frac{m_e^2}{E_e} \right) \right\}, \end{aligned} \quad (50)$$

where the coefficients a_j , A_j and B_j for $j = 1, 2, 3$ are defined in Eq.(11), Eq.(12) and Eq.(13), respectively, with the replacement $\lambda \rightarrow \lambda_{\text{eff}}$. The same replacement defines the correlation coefficients a_0 , A_0 and B_0 in terms of λ_{eff} (see Eq.(8)).

The correlation coefficients $a(E_e)$, $A(E_e)$ and $B(E_e)$, extended by the contributions of interactions beyond the SM, together with the correlation coefficients $K_n(E_e)$ and $Q_n(E_e)$, caused by the contributions of the $1/M$ corrections from the “weak magnetism” and the proton recoil only, determine the asymmetries $A_{\text{exp}}(E_e)$ and $B_{\text{exp}}(E_e)$. For the calculation of the electron asymmetry $A_{\text{exp}}(E_e)$, taking into account the contributions of interactions beyond the SM, we have to replace the correlation coefficient $A^{(W)}(E_e)$ in Eq.(17) by the expression

$$A^{(W)}(E_e) = A_0 \left(1 - b_F \frac{m_e}{E_e} \right) \left\{ 1 - \frac{1}{M} \frac{1}{2\lambda_{\text{eff}}(1 + \lambda_{\text{eff}})(1 + 3\lambda_{\text{eff}}^2)} \left(A_1^{(W)} E_0 + A_2^{(W)} E_e + A_3^{(W)} \frac{m_e^2}{E_e} \right) \right\}, \quad (51)$$

where the coefficients $A_j^{(W)}$ are given in Eq.(20) with the replacement $\lambda \rightarrow \lambda_{\text{eff}}$.

The antineutrino asymmetry $B_{\text{exp}}(E_e)$ is defined by Eq.(27) and Eq.(28) for $r \leq 1$ and $r \geq 1$, respectively. For the account for the contributions of interactions beyond the SM the correlation coefficients $a(E_e)$, $A(E_e)$ and $B(E_e)$ should be taken in the form, given by Eq.(50).

The proton recoil asymmetry C_{exp} completes the set of asymmetries, which can be measured in the neutron β^- -decay with a polarised neutron and unpolarised proton and electron. The correlation coefficient C_{eff} (see Appendix I), defining the asymmetry C_{exp} and extended by the contributions of interactions beyond the SM, takes the form

$$C_{\text{eff}} = -x_C \left(A_0 + B_0 \right) + \frac{\lambda(1 + \lambda_{\text{eff}})}{1 + 3\lambda_{\text{eff}}^2} b_F \frac{X_{17}}{X_1} - \frac{1}{2} \frac{(1 + \lambda_{\text{eff}})(1 - 3\lambda_{\text{eff}})}{1 + 3\lambda_{\text{eff}}^2} c_{ST} \frac{X_{18}}{X_1} + C_{\text{SM}}, \quad (52)$$

where $C_{\text{SM}} = C + x_C(A_0 + B_0)$ and $x_C = 27591$ (see Eq.(37)). The numerical factors $X_{17}/X_1 = -0.90187$ and $X_{18}/X_1 = 0.39806$ are calculated in Appendix I (see Eq.(I-22)).

Using the results, obtained in Appendix G, the theoretical expression for the rate of the neutron β^- -decay, including the contributions of interactions beyond the SM, is

$$\begin{aligned} \lambda_n &= (\lambda_n)_{\text{SM}} \left(1 + \frac{1}{1 + 3\lambda^2} \left(\text{Re}(\delta C_V - \delta \bar{C}_V) + 3\lambda \text{Re}(\delta C_A - \delta \bar{C}_A) \right) + b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right) = \\ &= \frac{G_F^2 |V_{ud}|^2}{2\pi^3} f_n(E_0, Z = 1) (1 + 3\lambda^2) \left(1 + \frac{1}{1 + 3\lambda^2} \left(\text{Re}(\delta C_V - \delta \bar{C}_V) - 3\lambda \text{Re}(\delta C_A - \delta \bar{C}_A) \right) + b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right), \end{aligned} \quad (53)$$

where $(\lambda_n)_{\text{SM}}$ is the lifetime of the neutron, calculated within the SM (see Eq.(39) and Eq.(40)), and $\langle m_e/E_e \rangle_{\text{SM}}$ is the average value, calculated with the electron-energy spectrum density Eq.(D-59). Now we have to define the rate of the neutron β^- -decay Eq.(53) in terms of the axial coupling constant λ_{eff} , which is related to the axial coupling constant λ as

$$\lambda = \lambda_{\text{eff}} + \frac{1}{2} (\lambda_{\text{eff}} \text{Re}(\delta C_V - \delta \bar{C}_V) + \text{Re}(\delta C_A - \delta \bar{C}_A)). \quad (54)$$

Substituting Eq.(54) into Eq.(53) and keeping only the linear terms in power of $\text{Re}(\delta C_V - \delta \bar{C}_V)$ and $\text{Re}(\delta C_A - \delta \bar{C}_A)$ one may show that

$$\begin{aligned} (1 + 3\lambda^2) \left(1 + \frac{1}{1 + 3\lambda^2} \left(\text{Re}(\delta C_V - \delta \bar{C}_V) - 3\lambda \text{Re}(\delta C_A - \delta \bar{C}_A) \right) + b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right) = \\ = \left(1 + \frac{1}{2} \text{Re}(\delta C_V - \delta \bar{C}_V) \right) (1 + 3\lambda_{\text{eff}}^2) \left(1 + b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right). \end{aligned} \quad (55)$$

This gives the rate of the neutron β^- -decay equal to

$$\lambda_n = \frac{G_F^2 |V_{ud}|^2}{2\pi^3} f_n(E_0, Z = 1) \left(1 + \frac{1}{2} \text{Re}(\delta C_V - \delta \bar{C}_V) \right) (1 + 3\lambda_{\text{eff}}^2) \left(1 + b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right), \quad (56)$$

where the Fermi integral is given by Eq.(40) with the replacement $\lambda \rightarrow \lambda_{\text{eff}}$.

If we introduce again $(\lambda_n)_{\text{SM}}$, defined by Eq.(39) and Eq.(40) with the replacement $\lambda \rightarrow \lambda_{\text{eff}}$, we get the following expression for the rate of the neutron β^- -decay, corrected by the contributions of interactions beyond the SM taken to linear approximation with respect to the Herczeg phenomenological coupling constants

$$\lambda_n = (\lambda_n)_{\text{SM}} \left(1 + \text{Re}(a_{LL}^h + a_{LR}^h) \right) \left(1 + b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right), \quad (57)$$

where we have set $\text{Re}(\delta C_V - \delta \bar{C}_V) = 2\text{Re}(a_{LL}^h + a_{LR}^h)$. Thus, at first glimpse a deviation of the experimental values $\tau_n = 1/\lambda_n$ of the lifetime of the neutron considered relative to the theoretical value of the lifetime of the neutron $(\tau_n)_{\text{SM}} = 1/(\lambda_n)_{\text{SM}}$, calculated in the SM at zero Herczeg coupling constants, may give an information about the contribution of the Herczeg left-left and left-right lepton-nucleon current-current interactions (vector and axial-vector interactions beyond the SM) by using the experimental value of the Fierz term b_F , determined from the experimental data on the electron-proton energy distribution $a(E_e, T_p)$, the proton-energy spectrum $a(T_p)$ and the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$ and C_{exp} . However, the problem is in the following that the Herczeg phenomenological interactions beyond the SM, when they exist, should exist always together with the interactions of the SM, and the decomposition of these interactions is rather artificial and based on the smallness ones with respect to others.

Hence, using the Hamilton Eq.(G-1) and Eq.(G-2) we may redefine the axial coupling constant and the CKM matrix element as follows $\lambda_{\text{eff}} = (\lambda - a_{LL}^h + a_{LR}^h)/(1 + a_{LL}^h + a_{LR}^h)$ and $(V_{ud})_{\text{eff}} = V_{ud}(1 + a_{LL}^h + a_{LR}^h)$ as it has been proposed in [43]–[45]. After such a change one may show that the rate of the neutron β^- -decay may contain only the contribution of the Fierz term

$$\lambda_n = (\lambda_n)_{\text{SM}} \left(1 + b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right), \quad (58)$$

where $(\lambda_n)_{\text{SM}}$ is given by Eq.(39) and Eq.(40) with the replacement $\lambda \rightarrow \lambda_{\text{eff}}$ and $V_{ud} \rightarrow (V_{ud})_{\text{eff}}$.

The definition of the effective coupling constant $\lambda_{\text{eff}} = (\lambda - a_{LL}^h + a_{LR}^h)/(1 + a_{LL}^h + a_{LR}^h)$ and the CKM matrix element $(V_{ud})_{\text{eff}} = V_{ud}(1 + a_{LL}^h + a_{LR}^h)$ at the Hamiltonian level introduces the imaginary parts to the axial coupling constant and the CKM matrix element

$$\begin{aligned}\lambda_{\text{eff}} &= \text{Re}\lambda_{\text{eff}} + i\text{Im}\lambda_{\text{eff}}, \\ (V_{ud})_{\text{eff}} &= V_{ud}(1 + \text{Re}(a_{LL}^h + a_{LR}^h))(1 + i\text{Im}(a_{LL}^h + a_{LR}^h))\end{aligned}\quad (59)$$

where $\text{Im}\lambda_{\text{eff}}$ is equal to

$$\text{Im}\lambda_{\text{eff}} = -(1 + \text{Re}\lambda_{\text{eff}})\text{Im}(a_{LL}^h) + (1 - \text{Re}\lambda_{\text{eff}})\text{Im}(a_{LR}^h). \quad (60)$$

An information about the imaginary part of the axial coupling constant λ_{eff} one may obtain by measuring the correlation coefficient $D(E_e)$, describing a violation of time reversal invariance. From Eq.(G-8) and Eq.(60) we obtain

$$\begin{aligned}D(E_e) &= D_{\text{SM}}(E_e) - \frac{2}{1 + 3\lambda_{\text{eff}}^2} \left(\lambda_{\text{eff}} \text{Im}(a_{LL}^h + a_{LR}^h) + \text{Im}(a_{LL}^h - a_{LR}^h) \right) = \\ &= D_{\text{SM}}(E_e) - \frac{2}{1 + 3\lambda_{\text{eff}}^2} \left((1 + \lambda_{\text{eff}})\text{Im}(a_{LL}^h) - (1 - \lambda_{\text{eff}})\text{Im}(a_{LR}^h) \right) = D_{\text{SM}}(E_e) + \frac{2\text{Im}\lambda_{\text{eff}}}{1 + 3\lambda_{\text{eff}}^2},\end{aligned}\quad (61)$$

where we have replaced $\text{Re}\lambda_{\text{eff}}$ by λ_{eff} , having neglected the contribution of the imaginary part, and $D_{\text{SM}}(E_e)$ is the contribution to the correlation coefficient $D(E_e)$, calculated within the SM [56]–[61]. As we have estimated in section III such a contribution, caused by the electron–proton interaction in the final state [56]–[59, 61], is of order 10^{-5} for the electron kinetic energies $250 \text{ keV} \leq T_e \leq 455 \text{ keV}$. Thus, the correlation coefficient $D(E_e)$, defined in the same energy region, should be sensitive to the contributions beyond the SM of 10^{-4} . Recently the experimental value $D_{\text{exp}}(E_e) = (-4 \pm 6) \times 10^{-4}$ of the correlation coefficient $D(E_e)$ [1, 9] has been substantially improved with a result $D_{\text{exp}}(E_e) = (-0.96 \pm 1.89_{\text{stat}} \pm 1.01_{\text{syst}}) \times 10^{-4}$ [88]. However, the new experimental value as well as the old one still implies that to order 10^{-4} the correlation coefficient $D(E_e)$ is commensurable with zero. Hence, to order 10^{-4} the imaginary part of the axial coupling constant λ_{eff} is also commensurable with zero, i.e. $\text{Im}\lambda_{\text{eff}} = 0$. Nevertheless, setting $\lambda_{\text{eff}} = \text{Re}\lambda_{\text{eff}} = -1.2750$ and using the relation $\text{Im}\lambda_{\text{eff}} = 0$ we may obtain that $\text{Im}(a_{LR}^h) = -0.12\text{Im}(a_{LL}^h)$. This adds an additional phase shift $e^{i0.88\text{Im}(a_{LL}^h)}$ to the CKM matrix element $(V_{ud})_{\text{eff}}$.

The axial coupling constant λ_a and the correlation coefficient a_0 may be also determined by measuring the proton–energy spectrum Eq.(33) (see [40]). The proton–energy spectrum, taking into account the contributions of interactions beyond the SM, is

$$a(T_p) = \left(1 - b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}}\right) \left\{ g_1(T_p) + a_0 \left(1 + \frac{1}{1 - \lambda^2} \frac{E_0}{M}\right) g_2(T_p) \right\}, \quad (62)$$

where the functions $g_1(T_p)$ and $g_2(T_p)$ are defined by the integrals

$$\begin{aligned}g_1(T_p) &= \int_{(E_e)_{\min}}^{(E_e)_{\max}} \left(1 + b_F \frac{m_e}{E_e}\right) \left(1 + \frac{\alpha}{\pi} g_n(E_e)\right) \zeta_1(E_e, T_p) F(E_e, Z=1) E_e dE_e, \\ g_2(T_p) &= \int_{(E_e)_{\min}}^{(E_e)_{\max}} \left(1 + \frac{\alpha}{\pi} g_n(E_e) + \frac{\alpha}{\pi} f_n(E_e)\right) \zeta_2(E_e, T_p) F(E_e, Z=1) E_e dE_e,\end{aligned}\quad (63)$$

where the limits of integration $(E_e)_{\max/\min}$ are given in Appendix I.

Summary. We have analysed the sensitivity of the electron–proton energy distribution $a(E_e, T_p)$, the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$, C_{exp} and the lifetime of the neutron τ_n to contributions of order 10^{-4} of interactions beyond the SM, taken to linear approximation with respect to the Herczeg phenomenological coupling constants of weak lepton–nucleon current–current interactions. We have shown that in such an approximation the axial coupling constant λ_{eff} and the CKM matrix element $(V_{ud})_{\text{eff}}$ absorb the contributions of the Herczeg left–left and left–right lepton–nucleon current–current interactions with the coupling constants a_{LL}^h and a_{LR}^h . In this approximation the axial coupling constant does not acquire an imaginary part to order 10^{-4} , but the CKM matrix element becomes an additional phase $e^{i\text{Im}(a_{LL}^h + a_{LR}^h)}$. Thus, after the measurements of the electron–proton energy spectrum $a(E_e, T_p)$, the proton–energy spectrum $a(T_p)$, the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$ and C_{exp} and the lifetime of the neutron τ_n one may determine the axial coupling constant λ_{eff} and the real parts of the scalar and tensor coupling constants a_S and a_T , defined in Eq.(45),

$$\begin{aligned}\text{Re}(a_S) + 3\lambda_{\text{eff}} \text{Re}(a_T) &= \frac{1 + 3\lambda_{\text{eff}}^2}{4} b_F \\ \lambda_{\text{eff}} \text{Re}(a_S) - \text{Re}(a_T) &= \frac{1 + 3\lambda_{\text{eff}}^2}{4} c_{ST},\end{aligned}\quad (64)$$

where in the r.h.s. of the algebraical equations λ_{eff} , b_F and c_{ST} are the experimental values with their experimental errors.

X. CONCLUSION

We have analysed the sensitivity of the electron–proton energy distribution $a(E_e, T_p)$, the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$ and C_{exp} of the correlations between the neutron spin and 3–momenta of the decay electron, antineutrino and proton, respectively, for the neutron β^- –decay with a polarised neutron and unpolarised proton and electron to contributions of order 10^{-4} of interactions beyond the SM. For the analysis of contributions of order 10^{-4} we have used the linear approximation for the correlation coefficient with respect to the Herczeg phenomenological coupling constants of weak lepton–nucleon current–current interactions. We have shown that in such an approximation the Herczeg right–left and right–right lepton–nucleon current–current interactions with the coupling constants a_{RL}^h and a_{RR}^h give no contributions to the correlation coefficients of the neutron β^- –decay and the lifetime of the neutron. Then, the contributions of the Herczeg left–left and left–right lepton–nucleon current–current interactions with the coupling constants a_{LL}^h and a_{LR}^h may be absorbed by the axial coupling constant, which we denote as $\lambda_{\text{eff}} = \lambda - \text{Re}(a_{LL}^h - a_{LR}^h) + \lambda \text{Re}(a_{LL}^h + a_{LR}^h)$ and the CKM matrix element $(V_{ud})_{\text{eff}} = V_{ud}(1 + \text{Re}(a_{LL}^h + a_{LR}^h))$. We have shown that the Herczeg coupling constants a_{LL}^h and a_{LR}^h in the effective Hamiltonian of weak interactions may be absorbed by the axial coupling constant $\lambda_{\text{eff}} = (\lambda - a_{LL}^h + a_{LR}^h)/(1 + a_{LL}^h + a_{LR}^h)$ and the CKM matrix element $(V_{ud})_{\text{eff}} = V_{ud}(1 + a_{LL}^h + a_{LR}^h)$. Using the experimental data on the correlation coefficient $D(E_e)$ we have shown that at the level of 10^{-4} the imaginary part of the axial coupling constant λ_{eff} is equal to zero, whereas the CKM matrix element acquires an additional phase $e^{i \text{Im}(a_{LL}^h + a_{LR}^h)}$. This shows that in addition to the background, calculated in the SM, the correlation coefficients of the neutron β^- –decay and the lifetime of the neutron, calculated to linear approximation with respect to the Herczeg coupling constants of lepton–nucleon current–current interactions, depend on the contributions of the scalar and tensor interactions only. This agrees with recent results, obtained in [43]–[45].

We have shown that the contributions of the scalar and tensor interactions beyond the SM are described by the Fierz term b_F and the coupling constant c_{ST} . The Fierz term may be determined from the experimental data on the asymmetry $A_{\text{exp}}(E_e)$ and the electron–proton energy distribution $a(E_e, T_p)$ (or the proton–energy spectrum $a(T_p)$). The coupling constant c_{ST} may be determined from the experimental data on the asymmetry $B_{\text{exp}}(E_e)$ and the proton recoil asymmetry C_{exp} . This allows to determine the scalar $\text{Re}(a_S)$ and tensor $\text{Re}(a_T)$ by solving the system of algebraical equations (see Eq.(64))

$$\begin{aligned} \text{Re}(a_S) + 3\lambda_{\text{eff}} \text{Re}(a_T) &= \frac{1 + 3\lambda_{\text{eff}}^2}{4} b_F \\ \lambda_{\text{eff}} \text{Re}(a_S) - \text{Re}(a_T) &= \frac{1 + 3\lambda_{\text{eff}}^2}{4} c_{ST}, \end{aligned}$$

The lifetime of the neutron is defined by the background, calculated in the SM, and the Fierz term (see Eq.(58))

$$\tau_n = (\tau_n)_{\text{SM}} \left(1 - b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right),$$

where $(\tau_n)_{\text{SM}}$ and $\langle m_e/E_e \rangle_{\text{SM}}$ are calculated in the SM.

We would like to note that the experimental analysis of the neutron β^- –decay with a polarised neutron and unpolarised proton and electron may be carried out in terms the electron–proton energy distribution $a(E_e, T_p)$, the proton–energy spectrum $a(T_p)$, the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$, C_{exp} and the lifetime of the neutron τ_n . From the fit of the experimental data on these energy distributions, asymmetries and the lifetime we determine three parameters, i.e. the axial coupling constant λ_{eff} , the Fierz term b_F and the coupling constant c_{ST} . In order to determine these parameters without correlations between them it suffices to use experimental data, obtained only in three from five independent experiments. This means that experimental data, obtained from other two independent experiments, should be described well by the parameters $(\lambda_{\text{eff}}, b_F, c_{ST})$, determined from the first three experiments. The deviations from the predicted values should be much smaller compared with 10^{-4} , since they may be explained only by the contributions of higher powers of the Herczeg coupling constants.

We have to note that the theoretical analysis of the sensitivity of the electron–proton energy distribution $a(E_e, T_p)$, the proton–energy spectrum $a(T_p)$, the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$ and C_{exp} of the neutron β^- –decay and the lifetime of the neutron to contributions of order 10^{-4} of interactions beyond the SM we have carried out above the background, calculated within the SM and the account for a complete set of the $1/M$ corrections, caused by the “weak magnetism” and the proton recoil, calculated to next–to–leading order in the large M or the large proton mass

expansion, and the radiative corrections of order α/π , calculated to leading order in the large M or the large proton mass expansion.

The corrections, caused by the “weak magnetism” and the proton recoil are calculated in analytical agreement with the results, obtained by Wilkinson [34] and Gudkov *et al.* [32]. The radiative corrections to the lifetime of the neutron and the correlation coefficients of the neutron β^- -decay with a polarised neutron and unpolarised decay proton and electron, calculated in this paper, we have given in terms of two functions $(\alpha/\pi)g_n(E_e)$ and $(\alpha/\pi)f_n(E_e)$. The analytical expressions of these functions are given in Eq.(D-58) of Appendix D. They are in analytical agreement with the radiative corrections, calculated in [18]–[30] and in [31, 32], respectively. We have confirmed Sirlin’s assertion that an unambiguous definition of the observable radiative corrections to the lifetime of the neutron is fully caused by the requirement of gauge invariance of the amplitude of one-virtual photon exchanges of the continuum-state β^- -decay of the neutron [18].

We have improved the theoretical expressions for the asymmetries $A_{\text{exp}}(E_e)$, $B_{\text{exp}}(E_e)$ and C_{exp} with respect to the expressions, used in [1, 37], [38, 39] and [42] for the experimental determination of the axial coupling constant $\lambda = -1.2750(9)$ and the correlation coefficients $A_0^{(\text{exp})} = -0.11933(34)$, $B_0^{(\text{exp})} = 0.9802(50)$ and $C^{(\text{exp})}_0 = -0.2377(26)$, respectively. We have added the radiative corrections to the electron asymmetry $A_{\text{exp}}(E_e)$ and the $1/M$ and radiative corrections to the antineutrino $B_{\text{exp}}(E_e)$ and proton C_{exp} asymmetries. In connection with the experimental analysis of contributions of order 10^{-4} of interactions beyond the SM to the neutron β^- -decay we have calculated the electron–proton energy distribution $a(E_e, T_p)$ and the proton–energy spectrum $a(T_p)$ by taking into account the complete set of the $1/M$ corrections, caused by the “weak magnetism” and the proton recoil, and the radiative corrections of order α/π .

We have also shown that at the present level of the experimental accuracy the lifetime of the neutron is described well by the SM with the account for a complete set of the $1/M$ corrections, caused by the “weak magnetism” and the proton recoil, calculated to next-to-leading order in the large proton mass expansion, and the radiative corrections of order α/π , calculated to leading order in the large proton mass expansion. The theoretical value of the lifetime of the neutron $(\tau_n)_{\text{SM}} = 879.6(1.1)\text{ s}$, where the error bars are defined by the error bars of the axial coupling constant $\lambda = -1.2750(9)$ and the CKM matrix element $V_{ud} = 0.97428(15)$, agrees well with the experimental values $\tau_n^{(\text{exp})} = 878.5(8)\text{ s}$, $\tau_n^{(\text{exp})} = 880.7(1.8)\text{ s}$ and $\tau_n^{(\text{exp})} = 881.6(2.1)\text{ s}$, measured by Serebrov *et al.* [12], Pilchmaier *et al.* [76] and Arzumanov *et al.* [77], respectively, and the world average values of the neutron lifetime $\tau_n^{(\text{w.a.v.})} = 880.1(1.1)\text{ s}$, $\tau_n^{(\text{w.a.v.})} = 880.0(9)\text{ s}$ and $\tau_n^{(\text{w.a.v.})} = 881.9(1.3)\text{ s}$, obtained in [9, 78, 79], respectively.

In order to demonstrate the sensitivity of the lifetime of the neutron, calculated in the SM, to the contributions of the radiative and $1/M$ corrections we propose to rewrite the Fermi integral $f_n(E_0, Z = 1)$, given by Eq.(40), in the following form

$$f_n(E_0, Z = 1) = \int_{m_e}^{E_0} (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) \left\{ \left(1 + k_1 \frac{\alpha}{\pi} g_n(E_e) \right) + k_2 \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left[\left(10\lambda^2 - 4(\kappa + 1)\lambda + 2 \right) E_e - 2\lambda \left(\lambda - (\kappa + 1) \right) \left(E_0 + \frac{m_e^2}{E_e} \right) \right] \right\} dE_e, \quad (65)$$

where the coefficients k_j for $j = 1, 2$ are equal to $k_j = 0$ or $k_j = 1$ that means that without $k_j = 0$ and with $k_j = 1$ corresponding corrections. The numerical values of the lifetime of the neutron for different k_j are adduced in Table I. It is seen that the most important contributions come from the radiative corrections.

| τ_n | k_1 | k_2 |
|----------|-------|-------|
| 915.3 s | 0 | 0 |
| 913.7 s | 0 | 1 |
| 881.0 s | 1 | 0 |
| 879.6 s | 1 | 1 |

TABLE I: The neutron lifetime, calculated for $\lambda = -1.2750$, the radiative corrections $k_1 = 0, 1$ and the $1/M$ corrections $k_2 = 0, 1$, caused by the “weak magnetism” and the proton recoil.

For the comparison of the theoretical lifetime of the neutron, defined by Eq.(39), with the expression, which is usually used for the measurement of the CKM matrix element V_{ud} [28] (see also [80]), we transcribe Eq.(39) into the form [28, 80]

$$\frac{1}{\tau_n} = C_n |V_{ud}|^2 (1 + 3\lambda^2) f(1 + \text{RC}), \quad (66)$$

where we have denoted $C_n = G_F^2 m_e^5 / 2\pi^3 = 1.1614 \times 10^{-4} \text{ s}^{-1}$ and $\text{RC} = \langle (\alpha/\pi) g_n(E_e) \rangle = 0.03886$, defining the radiative corrections [1, 28] integrated over the phase volume with the account for the proton–electron final-state

Coulomb interaction. Then, the phase-space factor f , including the $1/M$ corrections from the “weak magnetism” and the proton recoil, is determined by

$$f = \frac{1}{m_e^5} \int_{m_e}^{E_0} (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z=1) \left\{ 1 + \frac{1}{M} \frac{1}{1+3\lambda^2} \left[(10\lambda^2 - 4(\kappa+1)\lambda + 2) E_e - 2\lambda \left(\lambda - (\kappa+1) \right) \left(E_0 + \frac{m_e^2}{E_e} \right) \right] \right\} dE_e = 1.6894. \quad (67)$$

The numerical value agrees well with the value $f = 1.6887$, calculated in [28] (see also [80]). The factor $1 + \text{RC}$ we may represent in the following form $1 + \text{RC} = (1 + \delta_R)(1 + \Delta_R)$, where $\delta_R = \langle (\alpha/\pi) (g_n(E_e) - C_{WZ}) \rangle = 0.01505$ is defined by one-photon exchanges only [28, 29] and $\Delta_R = (\alpha/\pi) C_{WZ} = 0.02381$ is the part of the radiative corrections, induced by W -boson and Z -boson exchanges and QCD corrections [28, 29] (see also [32]). The phase-space factor f_R , including the contributions of the radiative corrections, caused by one-photon exchanges only, is equal to $f_R = f(1 + \delta_R) = 1.71483$. It does not contradict the value $f_R = 1.71385(34)$, used in [80] (see also [81]).

Currently the lifetime of the neutron is proposed to measuring in TU München within the project PENELOPE, using a superconducting magneto-gravitational trap of ultracold neutrons (UCN) for a precise neutron lifetime measurement [87]. In this experiment the UCN are trapped in a multipole field of a flux density up to 2 T and bound by a gravitational force at the top. This makes the extraction and detection of the protons possible and allows a direct measurement of neutron decay. A planing accuracy of 0.1 s and better demands high storage times and good knowledge of systematic errors, which could result from neutron spin flip and high energetic UCN which leave the storage volume only slowly. Therefore, the neutron spectrum is cleaned by an absorber. The big storage volume of 800 dm³ and the expected high neutron flux of FRMII give more than 10^7 neutrons per filling of the storage volume and meet statistical demands. Of course, the experimental data on the lifetime of the neutron, which should be obtained within this project with a planning accuracy better than 0.1 s, should place new constraints on contributions of interactions beyond the SM.

For the completeness of our analysis we have calculated (see Appendix H) the contributions of the proton recoil corrections of order α/M , caused by the electron-proton Coulomb interaction in the final state of the neutron β^- -decay. We have shown that these corrections to the lifetime of the neutron and the correlation coefficients are of order $10^{-6} - 10^{-5}$. This allows to neglect them for the analysis of contributions of order 10^{-4} of interactions beyond the SM.

We would like to note that we have used the experimental value of the axial coupling constant $\lambda = -1.2750(9)$, determined from the experimental data on the electron asymmetry $A_{\text{exp}}(E_e)$ [1, 37]. Such an experimental value of the axial coupling constant has been obtained with an unprecedented accuracy of about 0.07 %. The axial coupling constant $\lambda = -1.2750(9)$ agrees well with the axial coupling constants $\lambda = -1.2755(13)$, $\lambda = -1.2759_{-44.5}^{+40.9}$ and $\lambda = -1.2756(30)$, obtained recently by the PERKEO (PERKEO II) Collaboration [80] and the UCNA (Ultracold Neutron Asymmetry) Collaboration [82, 83], respectively, the accuracies of which are large compared with the accuracy of the axial coupling constant $\lambda = -1.2750(9)$. The lifetimes of the neutron $\tau_n = 879(2)$ s, $\tau_n = 879(6)$ s and $\tau_n = 879(4)$ s, calculated for the axial coupling constants $\lambda = -1.2755(13)$, $\lambda = -1.2579_{-44.5}^{+40.9}$ and $\lambda = -1.2756(30)$, respectively, agree with the experimental data $\tau_n^{(\text{exp})} = 878.5(8)$ s, $\tau_n^{(\text{exp})} = 880.7(1.8)$ s and $\tau_n^{(\text{exp})} = 881.6(2.1)$ s, measured by Serebrov *et al.* [12], Pilchmaier *et al.* [76] and Arzumanov *et al.* [77], respectively, and the world average values of the neutron lifetime $\tau_n^{(\text{w.a.v.})} = 880.1(1.1)$ s, $\tau_n^{(\text{w.a.v.})} = 880.0(9)$ s and $\tau_n^{(\text{w.a.v.})} = 881.9(1.3)$ s, obtained in [9, 78, 79], respectively.

The other experimental values of the axial coupling constant $\lambda = -1.266(4)$, $\lambda = -1.2594(38)$ and $\lambda = -1.262(5)$, obtained in [84, 85] and [86], respectively, and cited by [9], lead to the lifetimes of the neutron $\tau_n = 890(5)$ s, $\tau_n = 898(5)$ s and $\tau_n = 895(7)$ s, which do not agree with the world average values of the neutron lifetime $\tau_n^{(\text{w.a.v.})} = 880.1(1.1)$ s, $\tau_n^{(\text{w.a.v.})} = 880.0(9)$ s and $\tau_n^{(\text{w.a.v.})} = 881.9(1.3)$ s, obtained in [9, 78, 79], respectively. Moreover, the experimental methods, used in [84, 85] and [86] for the measurements of the electron asymmetry $A_{\text{exp}}(E_e)$, has been recently criticised in [80]. As has been pointed out by Mund *et al.* [80], in the experiments [84, 85] and [86] large corrections of about 15 % – 30 % have to be applied to 1) neutron polarisation, 2) magnetic mirror effects, 3) solid angle and 4) background.

For a long time [13]–[30] (see also [31, 32]) due to infrared divergences the calculation of the radiative β^- -decay of the neutron has been associated with the calculation of the radiative corrections to the neutron β^- -decay. As has been shown already in [13], the sum of the rates as well as the electron-energy and angular distributions of the continuum-state and radiative β^- -decay modes of the neutron does not suffer from infrared divergences, caused by one-virtual photon exchanges in the continuum-state β^- -decay mode and by the emission of real photons in radiative β^- -decay mode of the neutron.

Nevertheless, the radiative β^- -decay of the neutron $n \rightarrow p + e^- + \bar{\nu}_e + \gamma$ may be treated as a physical process, which may be observed separately from the neutron β^- -decay $n \rightarrow p + e^- + \bar{\nu}_e$. For the first time, the theoretical analysis

of the radiative β^- -decay of the neutron $n \rightarrow p + e^- + \bar{\nu}_e + \gamma$ as a physical observable process has been carried out in [49, 50]. First reliable experimental data on the branching ratio of the radiative β^- -decay of the neutron $\text{BR}_{\beta_c^- \gamma}^{(\text{exp})} = 3.13(35) \times 10^{-3}$, measured for the photon energy region $\omega_{\min} = 15 \text{ keV} \leq \omega \leq \omega_{\max} = 340 \text{ keV}$, have been reported by Nico *et al.* [46]. Then this result has been updated by Cooper *et al.* [47, 48], who have obtained $\text{BR}_{\beta_c^- \gamma}^{(\text{exp})} = 3.09(32) \times 10^{-3}$. These experimental values agree well with the theoretical value $\text{BR}_{\beta_c^- \gamma} = 2.85 \times 10^{-3}$, calculated by Gardner within HB χ PT for the same photon energy region [46, 47, 50]. In Appendix B we have carried out the calculation of the rate, the electron–photon energy and photon–energy spectra and angular distributions of the radiative β^- -decay of the neutron with a polarised neutron and unpolarised decay particles. Our results for the branching ratios $\text{BR}_{\beta_c^- \gamma} = 2.87 \times 10^{-3}$ and $\text{BR}_{\beta_c^- \gamma} = 4.45 \times 10^{-3}$, calculated for the photon energy regions $\omega_{\min} = 15 \text{ keV} \leq \omega \leq 350 \text{ keV}$ and $\omega_{\min} = 5 \text{ keV} \leq \omega \leq E_0 - m_e$, respectively, agree well with the results $\text{BR}_{\beta_c^- \gamma} = 2.85 \times 10^{-3}$ and $\text{BR}_{\beta_c^- \gamma} = 4.41 \times 10^{-3}$, obtained by Gardner [46, 47] and Bernard *et al.* [50], respectively. Within one standard deviation the branching ratio $\text{BR}_{\beta_c^- \gamma} = 2.87 \times 10^{-3}$ agrees also with the experimental data [46, 47].

The rate of the radiative β^- -decay of the neutron in dependence of a photon polarisation has been calculated in [50]. We argue that the more precise theoretical and experimental analysis of the energy spectra and angular distributions of the radiative β^- -decay of the neutron in dependence of the polarisations of the neutron and photon should be of great deal of importance for a test of the SM. We are planning to perform such a theoretical analysis in our forthcoming publication.

The theoretical analysis of the sensitivity of 1) the electron–proton energy distribution and the proton–energy spectrum, 2) the electron, antineutrino and proton asymmetries of correlations between the neutron spin and the 3-momenta of the decay particles and 3) the lifetime of the neutron is carried out according to the experimental program on the contract I534-N20 PERC, the theoretical program on the contract I689-N16, supported both by the Austrian “Fonds zur Förderung der Wissenschaftlichen Forschung” (FWF), and the experimental program of the experimental group of the Petersburg Nuclear Physics Institute (PNPI), headed by A. P. Serebrov, on the contract No. 11-02-91000 -ANF_a, supported by the Russian Foundation for Basic Research.

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Appendix A: Amplitude of continuum-state β^- -decay of neutron with “weak magnetism” and proton recoil corrections to order $1/M$

The amplitude of the continuum-state β^- -decay of the neutron we rewrite as follows

$$M(n \rightarrow pe^- \bar{\nu}_e) = -\frac{G_F}{\sqrt{2}} V_{ud} \mathcal{M}_{\beta_c^-}, \quad (\text{A-1})$$

where $\mathcal{M}_{\beta_c^-} = [\bar{u}_p O_\mu u_n][\bar{u}_e \gamma^\mu (1 - \gamma^5) v_{\bar{\nu}}]$ and the matrix O_μ takes the form

$$O_\mu = \gamma_\mu (1 + \lambda \gamma^5) + i \frac{\kappa}{2M} \sigma_{\mu\nu} (k_p - k_n)^\nu. \quad (\text{A-2})$$

In terms of the time and space components of the matrix $O_\mu = (O^0, -\vec{O})$ the amplitude $\mathcal{M}_{\beta_c^-}$ is defined by

$$\mathcal{M}_{\beta_c^-} = [\bar{u}_p O^0 u_n][\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - [\bar{u}_p \vec{O} u_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}]. \quad (\text{A-3})$$

The time O^0 and spacial \vec{O} components of the matrix O_μ we determine to first in the large M expansion. They read

$$O^0 = \begin{pmatrix} 1 & \lambda + \frac{\kappa}{2M} (\vec{\sigma} \cdot \vec{k}_p) \\ -\lambda + \frac{\kappa}{2M} (\vec{\sigma} \cdot \vec{k}_p) & -1 \end{pmatrix} \quad (\text{A-4})$$

and

$$\vec{O} = \begin{pmatrix} \lambda \vec{\sigma} + i \frac{\kappa}{2M} (\vec{\sigma} \times \vec{k}_p) & \vec{\sigma} \left(1 - \frac{\kappa}{2M} E_0\right) \\ -\vec{\sigma} \left(1 + \frac{\kappa}{2M} E_0\right) & -\lambda \vec{\sigma} + i \frac{\kappa}{2M} (\vec{\sigma} \times \vec{k}_p) \end{pmatrix}, \quad (\text{A-5})$$

where we have kept the terms of order $1/M$ only. For the calculation of the amplitude of the β^- -decay of the neutron we use the Dirac bispinorial wave functions of the neutron and the proton

$$u_n(\vec{0}, \sigma_n) = \sqrt{2m_n} \begin{pmatrix} \varphi_n \\ 0 \end{pmatrix}, \quad u_p(\vec{k}_p, \sigma_p) = \sqrt{E_p + m_p} \begin{pmatrix} \varphi_p \\ \frac{\vec{\sigma} \cdot \vec{k}_p}{E_p + m_p} \varphi_p \end{pmatrix}, \quad (\text{A-6})$$

where the Pauli spinorial wave functions φ_n and φ_p depend on the polarisations σ_n and σ_p , respectively. The matrix elements $[\bar{u}_p O^0 u_n]$ and $[\bar{u}_p \vec{O} u_n]$ are equal to

$$[\bar{u}_p O^0 u_n] = \sqrt{2m_n(E_p + m_p)} \left\{ [\varphi_p^\dagger \varphi_n] + \frac{\lambda}{2M} [\varphi_p^\dagger (\vec{\sigma} \cdot \vec{k}_p) \varphi_n] \right\} \quad (\text{A-7})$$

and

$$[\bar{u}_p \vec{O} u_n] = \sqrt{2m_n(E_p + m_p)} \left\{ \lambda [\varphi_p^\dagger \vec{\sigma} \varphi_n] + i \frac{\kappa}{2M} [\varphi_p^\dagger (\vec{\sigma} \times \vec{k}_p) \varphi_n] + \frac{1}{2M} [\varphi_p^\dagger (\vec{\sigma} \cdot \vec{k}_p) \vec{\sigma} \varphi_n] \right\}, \quad (\text{A-8})$$

where in curly brackets we have kept the contributions of the terms of order $1/M$ only. Using the relation $(\vec{\sigma} \cdot \vec{k}_p) \vec{\sigma} = \vec{k}_p + i (\vec{\sigma} \times \vec{k}_p)$ we rewrite the r.h.s. of Eq.(A-8) as follows

$$[\bar{u}_p \vec{O} u_n] = \sqrt{2m_n(E_p + m_p)} \left\{ \lambda [\varphi_p^\dagger \vec{\sigma} \varphi_n] + i \frac{\kappa + 1}{2M} [\varphi_p^\dagger (\vec{\sigma} \times \vec{k}_p) \varphi_n] + \frac{\vec{k}_p}{2M} [\varphi_p^\dagger \varphi_n] \right\}. \quad (\text{A-9})$$

Thus, the amplitude $\mathcal{M}_{\beta_c^-}$ is given by

$$\begin{aligned} \mathcal{M}_{\beta_c^-} &= \sqrt{2m_n(E_p + m_p)} \left\{ [\varphi_p^\dagger \varphi_n] [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - \lambda [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] + \frac{\lambda}{2M} [\varphi_p^\dagger (\vec{\sigma} \cdot \vec{k}_p) \varphi_n] \right. \\ &\quad \times [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - i \frac{\kappa + 1}{2M} [\varphi_p^\dagger (\vec{\sigma} \times \vec{k}_p) \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] - \frac{\vec{k}_p}{2M} [\varphi_p^\dagger \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] \left. \right\}. \end{aligned} \quad (\text{A-10})$$

For the transformation of the last term we use the following identity

$$-\frac{\vec{k}_p}{2M} [\varphi_p^\dagger \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] = \frac{E_0}{2M} [\varphi_p^\dagger \varphi_n] [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - \frac{m_e}{2M} [\varphi_p^\dagger \varphi_n] [\bar{u}_e (1 - \gamma^5) v_{\bar{\nu}}], \quad (\text{A-11})$$

based on the Dirac equation for the electron and antineutrino. Substituting Eq.(A-11) into Eq.(A-10) we obtain

$$\begin{aligned} \mathcal{M}_{\beta_c^-} &= \sqrt{2m_n(E_p + m_p)} \left\{ \left(1 + \frac{E_0}{2M}\right) [\varphi_p^\dagger \varphi_n] [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - \lambda [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] + \frac{\lambda}{2M} [\varphi_p^\dagger (\vec{\sigma} \cdot \vec{k}_p) \varphi_n] \right. \\ &\quad \times [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - i \frac{\kappa + 1}{2M} [\varphi_p^\dagger (\vec{\sigma} \times \vec{k}_p) \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] - \frac{m_e}{2M} [\varphi_p^\dagger \varphi_n] [\bar{u}_e (1 - \gamma^5) v_{\bar{\nu}}] \left. \right\}. \end{aligned} \quad (\text{A-12})$$

The next step of the calculation is to expand the normalisation factor $\sqrt{E_p + m_p}$ of the bispinorial wave function of the proton. This gives

$$\sqrt{2m_n(E_p + m_p)} = 2m_n \left(1 - \frac{E_0}{2M}\right), \quad (\text{A-13})$$

where we have kept the next-to-leading terms in the large M expansion.

Thus the amplitude $\mathcal{M}_{\beta_c^-}$, calculated to next-to-leading order in the large M expansion, is

$$\begin{aligned} \mathcal{M}_{\beta_c^-} = 2m_n \bigg\{ & [\varphi_p^\dagger \varphi_n] [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - \tilde{\lambda} [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] + \frac{\tilde{\lambda}}{2M} [\varphi_p^\dagger (\vec{\sigma} \cdot \vec{k}_p) \varphi_n] \\ & \times [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - i \frac{\kappa + 1}{2M} [\varphi_p^\dagger (\vec{\sigma} \times \vec{k}_p) \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] - \frac{m_e}{2M} [\varphi_p^\dagger \varphi_n] [\bar{u}_e (1 - \gamma^5) v_{\bar{\nu}}] \bigg\}, \end{aligned} \quad (\text{A-14})$$

where we have denoted $\tilde{\lambda} = \lambda(1 - E_0/2M)$. The hermitian conjugate amplitude takes the form

$$\begin{aligned} \mathcal{M}_{\beta_c^-}^\dagger = 2m_n \bigg\{ & [\varphi_n^\dagger \varphi_p] [\bar{v}_{\bar{\nu}} \gamma^0 (1 - \gamma^5) u_e] - \tilde{\lambda}^* [\varphi_n^\dagger \vec{\sigma} \varphi_p] \cdot [\bar{v}_{\bar{\nu}} \vec{\gamma} (1 - \gamma^5) u_e] + \frac{\tilde{\lambda}^*}{2M} [\varphi_n^\dagger (\vec{\sigma} \cdot \vec{k}_p) \varphi_p] \\ & \times [\bar{v}_{\bar{\nu}} \gamma^0 (1 - \gamma^5) u_e] + i \frac{\kappa + 1}{2M} [\varphi_n^\dagger (\vec{\sigma} \times \vec{k}_p) \varphi_p] \cdot [\bar{v}_{\bar{\nu}} \vec{\gamma} (1 - \gamma^5) u_e] - \frac{m_e}{2M} [\varphi_n^\dagger \varphi_p] [\bar{v}_{\bar{\nu}} (1 + \gamma^5) u_e] \bigg\}. \end{aligned} \quad (\text{A-15})$$

Substituting Eq.(A-14) into Eq.(A-1) we arrive at the amplitude of the continuum-state β^- -decay of the neutron, taking into account the contributions of the “weak magnetism” and the proton recoil, calculated to next-to-leading order in the $1/M$ expansion.

The electron-energy and angular distribution is proportional to $\frac{1}{2} \sum_{\text{pol}} |\mathcal{M}_{\beta_c^-}|^2$, where we sum over all polarisations of the interacting particles. Recall that the antineutrino is polarised in the direction parallel to its 3-momentum. The quantity $\frac{1}{2} \sum_{\text{pol}} |\mathcal{M}_{\beta_c^-}|^2$ is equal to

$$\begin{aligned} \sum_{\text{pol}} \frac{|\mathcal{M}_{\beta_c^-}|^2}{4m_n^2} = & \text{tr}\{1 + \vec{\xi}_n \cdot \vec{\sigma}\} \text{tr}\{(\hat{k}_e + m_e) \gamma^0 \hat{k} \gamma^0 (1 - \gamma^5)\} - \tilde{\lambda} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) \vec{\sigma}\} \cdot \text{tr}\{(\hat{k}_e + m_e) \vec{\gamma} \hat{k} \gamma^0 (1 - \gamma^5)\} \\ & - \tilde{\lambda}^* \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) \vec{\sigma}\} \cdot \text{tr}\{(\hat{k}_e + m_e) \gamma^0 \hat{k} \vec{\gamma} (1 - \gamma^5)\} + |\tilde{\lambda}|^2 \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) \sigma^i \sigma^j\} \text{tr}\{(\hat{k}_e + m_e) \gamma^j \hat{k} \gamma^i (1 - \gamma^5)\} \\ & - \frac{m_e}{2M} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma})\} \text{tr}\{(\hat{k}_e + m_e) \gamma^0 \hat{k} (1 + \gamma^5)\} - \frac{m_e}{2M} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma})\} \text{tr}\{(\hat{k}_e + m_e) \hat{k} \gamma^0 (1 - \gamma^5)\} \\ & + \tilde{\lambda} \frac{m_e}{2M} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) \vec{\sigma}\} \cdot \text{tr}\{(\hat{k}_e + m_e) \vec{\gamma} \hat{k} (1 + \gamma^5)\} + \tilde{\lambda}^* \frac{m_e}{2M} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) \vec{\sigma}\} \cdot \text{tr}\{(\hat{k}_e + m_e) \hat{k} \vec{\gamma} (1 - \gamma^5)\} \\ & + \frac{\tilde{\lambda}}{2M} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) (\vec{\sigma} \cdot \vec{k}_p)\} \text{tr}\{(\hat{k}_e + m_e) \gamma^0 \hat{k} \gamma^0 (1 - \gamma^5)\} \\ & + \frac{\tilde{\lambda}^*}{2M} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) (\vec{\sigma} \cdot \vec{k}_p)\} \text{tr}\{(\hat{k}_e + m_e) \gamma^0 \hat{k} \gamma^0 (1 - \gamma^5)\} \\ & - \frac{|\tilde{\lambda}|^2}{2M} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) (\vec{\sigma} \cdot \vec{k}_p) \vec{\sigma}\} \cdot \text{tr}\{(\hat{k}_e + m_e) \vec{\gamma} \hat{k} \gamma^0 (1 - \gamma^5)\} \\ & - \frac{|\tilde{\lambda}|^2}{2M} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) \vec{\sigma} (\vec{\sigma} \cdot \vec{k}_p)\} \cdot \text{tr}\{(\hat{k}_e + m_e) \gamma^0 \hat{k} \vec{\gamma} (1 - \gamma^5)\} \\ & - i \frac{\kappa + 1}{2M} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) (\vec{\sigma} \times \vec{k}_p)\} \cdot \text{tr}\{(\hat{k}_e + m_e) \vec{\gamma} \hat{k} \gamma^0 (1 - \gamma^5)\} \\ & + i \frac{\kappa + 1}{2M} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) (\vec{\sigma} \times \vec{k}_p)\} \cdot \text{tr}\{(\hat{k}_e + m_e) \gamma^0 \hat{k} \vec{\gamma} (1 - \gamma^5)\} \\ & + i \tilde{\lambda}^* \frac{\kappa + 1}{2M} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) \sigma^j (\vec{\sigma} \times \vec{k}_p)^\ell\} \text{tr}\{(\hat{k}_e + m_e) \gamma^\ell \hat{k} \gamma^j (1 - \gamma^5)\} \\ & - i \tilde{\lambda} \frac{\kappa + 1}{2M} \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) (\vec{\sigma} \times \vec{k}_p)^\ell \sigma^j\} \text{tr}\{(\hat{k}_e + m_e) \gamma^j \hat{k} \gamma^\ell (1 - \gamma^5)\}. \end{aligned} \quad (\text{A-16})$$

Since in our analysis the axial coupling constant λ is real, below we set $\lambda^* = \lambda$. Calculating the traces with the real

axial coupling constant we obtain the following result

$$\begin{aligned}
\sum_{\text{pol}} \frac{|\mathcal{M}_{\beta_c^-}|^2}{32m_n^2} &= (1 + 3\tilde{\lambda}^2) E_e E + (1 - \tilde{\lambda}^2) (\vec{k}_e \cdot \vec{k}) - 2\tilde{\lambda}(1 + \tilde{\lambda}) (\vec{\xi}_n \cdot \vec{k}_e) E - 2\tilde{\lambda}(1 - \tilde{\lambda}) (\vec{\xi}_n \cdot \vec{k}) E_e \\
&+ \frac{1}{M} \left\{ -m_e^2 E + \tilde{\lambda} m_e^2 (\vec{\xi}_n \cdot \vec{k}) - \tilde{\lambda} (\vec{\xi}_n \cdot \vec{k}_e + \vec{\xi}_n \cdot \vec{k}) (E_e E + \vec{k}_e \cdot \vec{k}) + \tilde{\lambda}^2 \left[(E_e - \vec{\xi}_n \cdot \vec{k}_e) (\vec{k}_e \cdot \vec{k} + E^2) \right. \right. \\
&+ (E + \vec{\xi}_n \cdot \vec{k}) (k_e^2 + \vec{k}_e \cdot \vec{k}) \left. \right] - (\kappa + 1) \left[(\vec{\xi}_n \cdot \vec{k}_e) (\vec{k}_e \cdot \vec{k} + E^2) - (\vec{\xi}_n \cdot \vec{k}) (k_e^2 + \vec{k}_e \cdot \vec{k}) \right] \\
&- 2(\kappa + 1) \tilde{\lambda} \left[E(k_e^2 + \vec{k}_e \cdot \vec{k}) - E_e (\vec{k}_e \cdot \vec{k} + E^2) \right] + 2(\kappa + 1) \tilde{\lambda} (\vec{\xi}_n \cdot \vec{k}_e + \vec{\xi}_n \cdot \vec{k}) E_e E \\
&- (\kappa + 1) \tilde{\lambda} \left[(\vec{\xi}_n \cdot \vec{k}_e) (\vec{k}_e \cdot \vec{k} + E^2) + (\vec{\xi}_n \cdot \vec{k}) (k_e^2 + \vec{k}_e \cdot \vec{k}) \right] \left. \right\}. \tag{A-17}
\end{aligned}$$

Taking into account the contribution of the phase volume Eq.(4) (see also Eq.(A-23)) and keeping only the terms of order $1/M$ we have

$$\begin{aligned}
\Phi_{\beta_c^-}(\vec{k}_e, \vec{k}) \sum_{\text{pol}} \frac{|\mathcal{M}_{\beta_c^-}|^2}{32m_n^2} &= (1 + 3\lambda^2) E_e E + (1 - \lambda^2) (\vec{k}_e \cdot \vec{k}) - 2\lambda(1 + \lambda) (\vec{\xi}_n \cdot \vec{k}_e) E - 2\lambda(1 - \lambda) (\vec{\xi}_n \cdot \vec{k}) E_e \\
&+ \frac{1}{M} \left\{ \left[-3\lambda^2 E_e E + \lambda^2 (\vec{k}_e \cdot \vec{k}) + \lambda(2\lambda + 1) (\vec{\xi}_n \cdot \vec{k}_e) E - \lambda(2\lambda - 1) (\vec{\xi}_n \cdot \vec{k}) E_e \right] E_0 + \left[3(1 + 3\lambda^2) E_e E \right. \right. \\
&+ 3(1 - \lambda^2) (\vec{k}_e \cdot \vec{k}) - 6\lambda(1 + \lambda) (\vec{\xi}_n \cdot \vec{k}_e) E - 6\lambda(1 - \lambda) (\vec{\xi}_n \cdot \vec{k}) E_e \left. \right] E_e + \left[-3(1 + 3\lambda^2) (\vec{k}_e \cdot \vec{k}) E_e - 3(1 - \lambda^2) \right. \\
&\times \frac{(\vec{k}_e \cdot \vec{k})^2}{E} + 6\lambda(1 + \lambda) (\vec{\xi}_n \cdot \vec{k}_e) (\vec{k}_e \cdot \vec{k}) + 6\lambda(1 - \lambda) (\vec{\xi}_n \cdot \vec{k}) (\vec{k}_e \cdot \vec{k}) \frac{E_e}{E} \left. \right] - m_e^2 E + \lambda m_e^2 (\vec{\xi}_n \cdot \vec{k}) \\
&- \lambda (\vec{\xi}_n \cdot \vec{k}_e + \vec{\xi}_n \cdot \vec{k}) (E_e E + \vec{k}_e \cdot \vec{k}) + \lambda^2 \left[(E_e - \vec{\xi}_n \cdot \vec{k}_e) (\vec{k}_e \cdot \vec{k} + E^2) + (E + \vec{\xi}_n \cdot \vec{k}) (k_e^2 + \vec{k}_e \cdot \vec{k}) \right] \\
&- (\kappa + 1) \left[(\vec{\xi}_n \cdot \vec{k}_e) (\vec{k}_e \cdot \vec{k} + E^2) - (\vec{\xi}_n \cdot \vec{k}) (k_e^2 + \vec{k}_e \cdot \vec{k}) \right] - 2(\kappa + 1) \lambda \left[E(k_e^2 + \vec{k}_e \cdot \vec{k}) - E_e (\vec{k}_e \cdot \vec{k} + E^2) \right] \\
&+ 2(\kappa + 1) \lambda (\vec{\xi}_n \cdot \vec{k}_e + \vec{\xi}_n \cdot \vec{k}) E_e E - (\kappa + 1) \lambda \left[(\vec{\xi}_n \cdot \vec{k}_e) (\vec{k}_e \cdot \vec{k} + E^2) + (\vec{\xi}_n \cdot \vec{k}) (k_e^2 + \vec{k}_e \cdot \vec{k}) \right] \left. \right\}. \tag{A-18}
\end{aligned}$$

The function $\Phi_{\beta_c^-}(\vec{k}_e, \vec{k})$ is defined by the integral over the antineutrino energy E

$$\Phi_{\beta_c^-}(\vec{k}_e, \vec{k}) = \int_0^\infty \delta(f(E)) \frac{m_n}{E_p} \frac{E^2 dE}{(E_0 - E_e)^2}, \tag{A-19}$$

where the function $f(E)$ is $f(E) = m_n - E_p - E_e - E$ and $E_p = \sqrt{m_p^2 + (\vec{k}_e + \vec{k})^2}$ is the proton energy after the integration over the 3-momentum of the proton, giving $\vec{k}_p = -\vec{k}_e - \vec{k}$. Using the properties of the δ -function the result of the integration over E is equal to

$$\Phi_{\beta_c^-}(\vec{k}_e, \vec{k}) = \frac{m_n}{E_p} \frac{E^2}{(E_0 - E_e)^2} \frac{1}{\left| \frac{df(E)}{dE} \right|} \Big|_{E=E_r}, \tag{A-20}$$

where E_r is the root of the equation $f(E_r) = 0$. To next-to-leading order in the large M expansion the root E_r is equal to

$$E_r = (E_0 - E_e) \left(1 + \frac{1}{M} (E_e - k_e \cos \vartheta_{e\bar{\nu}}) \right), \tag{A-21}$$

where $\cos \vartheta_{e\bar{\nu}} = \vec{k}_e \cdot \vec{k} / k_e E$. Since

$$\frac{m_n}{E_p} \frac{1}{\left| \frac{df(E)}{dE} \right|} \Big|_{E=E_r} = \frac{m_n}{m_n - E_e + k_e \cos \vartheta_{e\bar{\nu}}} = 1 + \frac{1}{M} (E_e - k_e \cos \vartheta_{e\bar{\nu}}), \tag{A-22}$$

where we have kept the terms of order $1/M$ only, using Eq.(A-21) and Eq.(A-22) for the function $\Phi_{\beta_c^-}(\vec{k}_e, \vec{k})$ we obtain the following expression

$$\Phi_{\beta_c^-}(\vec{k}_e, \vec{k}) = 1 + \frac{3}{M} (E_e - k_e \cos \vartheta_{e\bar{\nu}}) = 1 + \frac{3}{M} \left(E_e - \frac{\vec{k}_e \cdot \vec{k}}{E} \right). \tag{A-23}$$

We have used this expression for the calculation of Eq.(A-18), which defines the correlation coefficients of the continuum-state β^- -decay of the neutron, calculated to next-to-leading order in the large M expansion (see Eq.(6)).

Appendix B: Radiative β^- -decay of neutron

In this Appendix we calculate the amplitude, the rate, the photon-electron and photon energy and angular distributions of the radiative β^- -decay $n \rightarrow p + e^- + \bar{\nu}_e + \gamma$ of the neutron. The Hamilton operator of weak interactions is defined by Eq.(1), whereas the Hamilton operator of electromagnetic interactions is

$$\mathcal{H}_{\text{em}}(x) = e [\bar{\psi}_p(x) \gamma^\mu \psi_p(x) - \bar{\psi}_e(x) \gamma^\mu \psi_e(x)] A_\mu(x), \quad (\text{B-1})$$

where e is the electric charge of the proton and $A_\mu(x) = (0, -\vec{A}(x))$ is the electromagnetic vector potential, taken in the Coulomb gauge $\text{div} \vec{A}(x) = 0$ [35].

For the calculation of the amplitude of the radiative β^- -decay $n \rightarrow p + e^- + \bar{\nu}_e + \gamma$ we take into account the contributions of the intermediate proton and electron states and drop the “weak magnetism” and proton recoil corrections. This gives

$$\begin{aligned} M(n \rightarrow p e^- \bar{\nu}_e \gamma) &= e \frac{G_F}{\sqrt{2}} V_{ud} [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] \frac{1}{2q \cdot k_e} [\bar{u}_e (2\varepsilon^* \cdot k_e + \hat{\varepsilon}^* \hat{q}) \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}] \\ &- e \frac{G_F}{\sqrt{2}} V_{ud} \frac{1}{2q \cdot k_p} [\bar{u}_p (2\varepsilon^* \cdot k_p + \hat{\varepsilon}^* \hat{q}) \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}], \end{aligned} \quad (\text{B-2})$$

where $\varepsilon = (0, \vec{\varepsilon})$ and $q = (\omega, \vec{q})$ are the polarisation vector and 4-momentum of the photon, obeying the constraint $\varepsilon \cdot q = 0$.

We calculate the amplitude of the radiative β^- -decay of the neutron to leading order in the large proton mass (or large M) expansion. This agrees well with 1) the neglect of the contributions of order $(\alpha/\pi)(E_0/M) \sim 10^{-6}$ (see the discussion below Eq.(20)) and 2) the assertion [47, 48], that the electron-photon energy spectrum, the photon polarisation observables and the rate of the radiative β^- -decay of the neutron are dominated by the electron emission of photons. In such an approximation the amplitude of the radiative β^- -decay takes the form

$$M(n \rightarrow p e^- \bar{\nu}_e \gamma) = e \frac{G_F}{\sqrt{2}} V_{ud} \frac{m_n}{\omega} \frac{\mathcal{M}_{\beta_c^- \gamma}}{E_e - \vec{n} \cdot \vec{k}_e}, \quad (\text{B-3})$$

where $\vec{n} = \vec{q}/\omega$. The amplitude $\mathcal{M}_{\beta_c^- \gamma}$ and its hermitian conjugate are determined by

$$\mathcal{M}_{\beta_c^- \gamma} = [\varphi_p^\dagger \varphi_n] [\bar{u}_e Q \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - \lambda [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e Q \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}], \quad (\text{B-4})$$

and

$$\mathcal{M}_{\beta_c^- \gamma}^\dagger = [\varphi_n^\dagger \varphi_p] [\bar{v}_{\bar{\nu}} \gamma^0 \bar{Q} (1 - \gamma^5) u_e] - \lambda [\varphi_n^\dagger \vec{\sigma} \varphi_p] \cdot [\bar{v}_{\bar{\nu}} \vec{\gamma} \bar{Q} (1 - \gamma^5) u_e], \quad (\text{B-5})$$

where $Q = 2(\varepsilon^* \cdot k_e) + \hat{\varepsilon}^* \hat{q}$ and $\bar{Q} = \gamma^0 Q^\dagger \gamma^0 = 2(\varepsilon \cdot k_e) + \hat{q} \hat{\varepsilon}$.

The squared absolute value of the amplitude Eq.(B-4), summed up over the polarisations of the proton and the electron in the final state accounting for the polarisation of the neutron is given by

$$\begin{aligned} \sum_{\text{pol.}} |\mathcal{M}_{\beta_c^- \gamma}|^2 &= \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma})\} \text{tr}\{\hat{k}_e Q \gamma^0 \hat{k} \gamma^0 \bar{Q} (1 - \gamma^5)\} - \lambda \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) \vec{\sigma}\} \cdot \text{tr}\{\hat{k}_e Q \gamma^0 \hat{k} \vec{\gamma} \bar{Q} (1 - \gamma^5)\} \\ &- \lambda \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) \vec{\sigma}\} \cdot \text{tr}\{\hat{k}_e Q \vec{\gamma} \hat{k} \gamma^0 \bar{Q} (1 - \gamma^5)\} + \lambda^2 \text{tr}\{(1 + \vec{\xi}_n \cdot \vec{\sigma}) \sigma^m \sigma^n\} \text{tr}\{\hat{k}_e Q \gamma^n \hat{k} \gamma^m \bar{Q} (1 - \gamma^5)\}. \end{aligned} \quad (\text{B-6})$$

Calculating the traces over the nucleon degrees of freedom and using the properties of the Dirac matrices

$$\gamma^\alpha \gamma^\nu \gamma^\mu = \gamma^\alpha g^{\nu\mu} - \gamma^\nu g^{\mu\alpha} + \gamma^\mu g^{\alpha\nu} + i \varepsilon^{\alpha\nu\mu\beta} \gamma_\beta \gamma^5, \quad (\text{B-7})$$

where $\varepsilon^{\alpha\nu\mu\beta}$ is the Levi-Civita tensor defined by $\varepsilon^{0123} = 1$ and $\varepsilon_{\alpha\nu\mu\beta} = -\varepsilon^{\alpha\nu\mu\beta}$ [35], we transcribe the r.h.s. of Eq.(B-6) into the form

$$\begin{aligned} \sum_{\text{pol.}} |\mathcal{M}_{\beta_c^- \gamma}|^2 &= 2E \left[\left((1 + 3\lambda^2) - 2\lambda(1 - \lambda) \frac{\vec{\xi}_n \cdot \vec{k}}{E} \right) \text{tr}\{\hat{k}_e Q \gamma^0 \bar{Q} (1 - \gamma^5)\} \right. \\ &\quad \left. + \left((1 - \lambda^2) \frac{\vec{k}}{E} - 2\lambda(1 + \lambda) \vec{\xi}_n \right) \cdot \text{tr}\{\hat{k}_e Q \vec{\gamma} \bar{Q} (1 - \gamma^5)\} \right]. \end{aligned} \quad (\text{B-8})$$

The traces in Eq.(B-8) are equal to

$$\begin{aligned} \frac{1}{16} \text{tr}\{\hat{k}_e Q \gamma^\mu \bar{Q} (1 - \gamma^5)\} &= (\varepsilon^* \cdot k_e)(\varepsilon \cdot k_e) (k_e + q)^\mu - \frac{1}{2} (\varepsilon^* \cdot \varepsilon) \left((k_e \cdot q) q^\mu - \frac{1}{2} q^2 k_e^\mu \right) \\ &- \frac{1}{2} \left(\varepsilon^* \cdot k_e \right) \varepsilon^\mu + (\varepsilon \cdot k_e) \varepsilon^{*\mu} \left(k_e \cdot q + \frac{1}{2} q^2 \right) + \frac{1}{2} i \varepsilon^{\mu\alpha\beta\nu} \left((\varepsilon^* \cdot k_e) \varepsilon_\alpha - (\varepsilon \cdot k_e) \varepsilon_\alpha^* \right) q_\beta k_{e\nu} \\ &+ \frac{1}{2} i \left(q^\mu q_\rho - \frac{1}{2} q^2 g_\rho^\mu \right) \varepsilon^{\rho\alpha\beta\nu} \varepsilon_\alpha^* \varepsilon_\beta k_{e\nu}, \end{aligned} \quad (\text{B-9})$$

where $q^2 = 0$ for a real transverse photon. Summing up over the photon polarisations we obtain the following photon–electron energy and angular distribution of the radiative β^- -decay of the neutron

$$\begin{aligned} \frac{d^8 \lambda_{\beta_c^- \gamma}(E_e, \omega, \vec{k}_e, \vec{k}, \vec{q}, \vec{\xi}_n)}{d\omega dE_e d\Omega_{\vec{k}_e} d\Omega_{\vec{k}} d\Omega_{\vec{n}}} &= \frac{\alpha}{2\pi} \frac{G_F^2 |V_{ud}|^2}{(2\pi)^6} (1 + 3\lambda^2) \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) (E_0 - E_e - \omega)^2 \frac{1}{\omega} \\ &\times \left\{ \left(1 + B_0 \frac{\vec{\xi}_n \cdot \vec{k}}{E} \right) \left(\frac{k_e^2 - (\vec{n} \cdot \vec{k}_e)^2}{(E_e - \vec{n} \cdot \vec{k}_e)^2} \left(1 + \frac{\omega}{E_e} \right) + \frac{1}{E_e - \vec{n} \cdot \vec{k}_e} \frac{\omega^2}{E_e} \right) \right. \\ &\left. + \left(a_0 \frac{\vec{k}}{E} + A_0 \vec{\xi}_n \right) \cdot \left[\left(\frac{k_e^2 - (\vec{n} \cdot \vec{k}_e)^2}{(E_e - \vec{n} \cdot \vec{k}_e)^2} + \frac{\omega}{E_e - \vec{n} \cdot \vec{k}_e} \right) \frac{\vec{k}_e}{E_e} + \left(-\frac{m_e^2}{(E_e - \vec{n} \cdot \vec{k}_e)^2} + \frac{E_e + \omega}{E_e - \vec{n} \cdot \vec{k}_e} \right) \frac{\vec{q}}{E_e} \right] \right\}. \end{aligned} \quad (\text{B-10})$$

For the unpolarised neutron Eq.(B-10) coincides with the spectrum, adduced in [47] (see Eq.(3) of Ref.[47] and a comment in [89]).

After the integration over the directions of the photon momentum the photon–electron energy and angular distribution takes the form

$$\begin{aligned} \frac{d^6 \lambda_{\beta_c^- \gamma}(E_e, \omega, \vec{k}_e, \vec{k}, \vec{q}, \vec{\xi}_n)}{d\omega dE_e d\Omega_{\vec{k}_e} d\Omega_{\vec{k}}} &= \frac{\alpha}{\pi} (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{(2\pi)^5} \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) (E_0 - E_e - \omega)^2 \frac{1}{\omega} \\ &\times \left\{ \left(1 + B_0 \frac{\vec{\xi}_n \cdot \vec{k}}{E} \right) \left\{ \left(1 + \frac{\omega}{E_e} + \frac{1}{2} \frac{\omega^2}{E_e^2} \right) \left[\frac{1}{\beta} \ln \left(\frac{1+\beta}{1-\beta} \right) - 2 \right] + \frac{\omega^2}{E_e^2} \right\} \right. \\ &\left. + \left(a_0 \frac{\vec{k} \cdot \vec{k}_e}{E E_e} + A_0 \frac{\vec{\xi}_n \cdot \vec{k}_e}{E_e} \right) \left[1 + \frac{1}{\beta^2} \frac{\omega}{E_e} \left(1 + \frac{1}{2} \frac{\omega}{E_e} \right) \right] \left[\frac{1}{\beta} \ln \left(\frac{1+\beta}{1-\beta} \right) - 2 \right] \right\}. \end{aligned} \quad (\text{B-11})$$

Integrating over the phase volume of the final state we obtain the rate of the radiative β^- -decay of the neutron

$$\begin{aligned} \lambda_{\beta_c^- \gamma}(\omega_{\min}) &= \frac{\alpha}{\pi} (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{2\pi^3} \int_{m_e}^{E_0 - \omega_{\min}} dE_e \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) \\ &\times \int_{\omega_{\min}}^{E_0 - E_e} \frac{d\omega}{\omega} (E_0 - E_e - \omega)^2 \left\{ \left(1 + \frac{\omega}{E_e} + \frac{1}{2} \frac{\omega^2}{E_e^2} \right) \left[\frac{1}{\beta} \ln \left(\frac{1+\beta}{1-\beta} \right) - 2 \right] + \frac{\omega^2}{E_e^2} \right\}. \end{aligned} \quad (\text{B-12})$$

The lowest photon energy ω_{\min} may be treated as the photon energy threshold of the detector.

For the photon energy interval $\omega_{\min} \leq \omega \leq \omega_{\max}$ the rate of the radiative β^- -decay of the neutron reads

$$\begin{aligned} \lambda_{\beta_c^- \gamma}(\omega_{\max}, \omega_{\min}) &= \frac{\alpha}{\pi} (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{2\pi^3} \int_{\omega_{\min}}^{\omega_{\max}} \frac{d\omega}{\omega} \int_{m_e}^{E_0 - \omega} dE_e \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) (E_0 - E_e - \omega)^2 \\ &\times \left\{ \left(1 + \frac{\omega}{E_e} + \frac{1}{2} \frac{\omega^2}{E_e^2} \right) \left[\frac{1}{\beta} \ln \left(\frac{1+\beta}{1-\beta} \right) - 2 \right] + \frac{\omega^2}{E_e^2} \right\}. \end{aligned} \quad (\text{B-13})$$

For the region of photon energies $15 \text{ keV} \leq \omega \leq 340 \text{ keV}$, used in the experiments by [46, 47], the branching ratio is equal to $\text{BR}_{\beta_c^- \gamma} = 2.87 \times 10^{-3}$. This result agrees well with the experimental values $\text{BR}_{\beta_c^- \gamma} = 3.13(35) \times 10^{-3}$ and $\text{BR}_{\beta_c^- \gamma} = 3.09(32) \times 10^{-3}$ and the result $\text{BR}_{\beta_c^- \gamma} = 2.85 \times 10^{-3}$, calculated by Gardner [46, 47].

For the comparison with the analysis of the radiative β^- -decay, carried out by Bernard *et al.* [50], we calculate the branching ratio for the photon energy region $5 \text{ keV} \leq \omega \leq E_0 - m_e$. The result $\text{BR}_{\beta_c^- \gamma} = 4.45 \times 10^{-3}$ is in a agreement with $\text{BR}_{\beta_c^- \gamma} = 4.41 \times 10^{-3}$, calculated in [50].

For the electron–energy and angular distribution of the radiative β^- –decay of the neutron we obtain the following expression

$$\begin{aligned} \frac{d^5 \lambda_{\beta^- \gamma}(E_e, \vec{k}_e, \vec{k}, \vec{\xi}_n, \omega_{\min})}{dE_e d\Omega_{\vec{k}_e} d\Omega_{\vec{k}}} &= \frac{\alpha}{\pi} (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{(2\pi)^5} (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} E_e F(E_e, Z = 1) \\ &\times \left\{ \left(1 + B_0 \frac{\vec{\xi}_n \cdot \vec{k}}{E}\right) g_{\beta^- \gamma}^{(1)}(E_e, \omega_{\min}) + \left(a_0 \frac{\vec{k} \cdot \vec{k}_e}{E E_e} + A_0 \frac{\vec{\xi}_n \cdot \vec{k}_e}{E_e}\right) g_{\beta^- \gamma}^{(2)}(E_e, \omega_{\min}) \right\}, \end{aligned} \quad (\text{B-14})$$

where the functions $g_{\beta^- \gamma}^{(1)}(E_e, \omega_{\min})$ and $g_{\beta^- \gamma}^{(2)}(E_e, \omega_{\min})$ are defined by

$$\begin{aligned} g_{\beta^- \gamma}^{(1)}(E_e, \omega_{\min}) &= \int_{\omega_{\min}}^{E_0 - E_e} \frac{d\omega}{\omega} \frac{(E_0 - E_e - \omega)^2}{(E_0 - E_e)^2} \left\{ \left(1 + \frac{\omega}{E_e} + \frac{1}{2} \frac{\omega^2}{E_e^2}\right) \left[\frac{1}{\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 2\right] + \frac{\omega^2}{E_e^2} \right\} = \\ &= \left[2\ell n\left(\frac{E_0 - E_e}{\omega_{\min}}\right) - 3 + \frac{2}{3} \frac{E_0 - E_e}{E_e} \left(1 + \frac{1}{8} \frac{E_0 - E_e}{E_e}\right)\right] \left[\frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 1\right] + \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2}, \\ g_{\beta^- \gamma}^{(2)}(E_e, \omega_{\min}) &= \int_{\omega_{\min}}^{E_0 - E_e} \frac{d\omega}{\omega} \frac{(E_0 - E_e - \omega)^2}{(E_0 - E_e)^2} \left[1 + \frac{1}{\beta^2} \frac{\omega}{E_e} \left(1 + \frac{1}{2} \frac{\omega}{E_e}\right)\right] \left[\frac{1}{\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 2\right] = \\ &= \left[2\ell n\left(\frac{E_0 - E_e}{\omega_{\min}}\right) - 3 + \frac{2}{3} \frac{E_0 - E_e}{\beta^2 E_e} \left(1 + \frac{1}{8} \frac{E_0 - E_e}{E_e}\right)\right] \left[\frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 1\right]. \end{aligned} \quad (\text{B-15})$$

We would like to note that as we show below the functions $g_{\beta^- \gamma}^{(1)}(E_e, \omega_{\min})$ and $g_{\beta^- \gamma}^{(2)}(E_e, \omega_{\min})$, calculated by means of the infrared cut–off regularisation ω_{\min} , have energy dependencies different in comparison with the functions, calculated in the FPM regularisation (see also [25]).

The discrepancy between the electron–energy and angular distributions, obtained for a real photon emission with the infrared cut–off and FPM regularisation, respectively, we discuss first in terms of the logarithmically divergent integral, which defines the infrared divergent contribution to the amplitude of the radiative β^- –decay of the neutron. This integral is

$$J(\beta) = \int \frac{d^3 q}{4\pi q_0} \frac{\beta^2 - (\vec{v} \cdot \vec{\beta})^2}{(q_0 - \vec{q} \cdot \vec{\beta})^2}, \quad (\text{B-16})$$

where $q_0 = \sqrt{\vec{q}^2 + \mu^2}$ is an energy of a photon with mass μ , $\vec{\beta} = \vec{k}_e/E_e$ and $\vec{v} = \vec{q}/q_0$ are the velocities of the electron and massive photon, respectively. The region of the integration in Eq.(B-16) is restricted by $q_1 \leq q \leq q_2$, where $q = |\vec{q}|$.

For the first time the integral Eq.(B-16) has been discussed by Kinoshita and Sirlin in [15]. It is obvious that replacing the lower and upper limits of the integral by ω_{\min} and $E_0 - E_e$, respectively, and setting μ zero in the integrand the function $J(\beta)$, given by

$$J(\beta) = 2\ell n\left(\frac{E_0 - E_e}{\omega_{\min}}\right) \left[\frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 1\right], \quad (\text{B-17})$$

defines the infrared divergent part of the functions $g_{\beta^- \gamma}^{(1)}(E_e, \omega_{\min})$ and $g_{\beta^- \gamma}^{(2)}(E_e, \omega_{\min})$ in Eq.(B-15).

Now let us calculate the integral Eq.(B-16) for $\mu \neq 0$. Following Kinoshita and Sirlin [15], we may rewrite the integral Eq.(B-16) as follows

$$J(\beta; q_2, q_1) = \frac{\beta^2}{2} \int_{-1}^{+1} dx \int_{q_1}^{q_2} \frac{dq q^2}{q_0^3} \frac{1 - v^2 x^2}{(1 - \beta v x)^2}. \quad (\text{B-18})$$

Making a change of variables $q \rightarrow v$, proposed by Kinoshita and Sirlin [15], we arrive at the integral

$$J(\beta; q_2, q_1) = \frac{\beta^2}{2} \int_{-1}^{+1} dx \int_{v_1}^{v_2} \frac{dv v^2}{1 - v^2} \frac{1 - v^2 x^2}{(1 - \beta v x)^2}. \quad (\text{B-19})$$

Integrating over x

$$J(\beta; q_2, q_1) = \int_{v_1}^{v_2} \left[1 + \frac{1}{1 - \beta^2 v^2} - \frac{2}{1 - v^2} + \frac{1}{\beta} \frac{v}{1 - v^2} \ln\left(\frac{1 + \beta v}{1 - \beta v}\right)\right] dv \quad (\text{B-20})$$

and over v we obtain

$$J(\beta; q_2, q_1) = \ell n \left[\frac{(1-v_1)(1+v_2)}{(1-v_2)(1+v_1)} \right] \left[\frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) - 1 \right] + (v_2 - v_1) + \frac{1}{2\beta} \ell n \left[\frac{(1+\beta v_2)(1-\beta v_1)}{(1-\beta v_2)(1+\beta v_1)} \right] \\ - \frac{1}{2\beta} L \left(\frac{\beta}{1+\beta} (1-v_2) \right) + \frac{1}{2\beta} L \left(\frac{\beta}{1+\beta} (1-v_1) \right) + \frac{1}{2\beta} L \left(-\frac{\beta}{1-\beta} (1-v_2) \right) - \frac{1}{2\beta} L \left(-\frac{\beta}{1-\beta} (1-v_1) \right) \\ + \frac{1}{2\beta} L \left(\frac{\beta}{1+\beta} (1+v_2) \right) - \frac{1}{2\beta} L \left(\frac{\beta}{1+\beta} (1+v_1) \right) - \frac{1}{2\beta} L \left(-\frac{\beta}{1-\beta} (1+v_2) \right) + \frac{1}{2\beta} L \left(-\frac{\beta}{1-\beta} (1+v_1) \right), \quad (\text{B-21})$$

where $L(x)$ is the Spence function, defined by [15, 90]

$$L(x) = \int_0^x \frac{dt}{t} \ell n |1-t|. \quad (\text{B-22})$$

For $q_1 \gg \mu$ the function $J(\beta)$ reduces to the form

$$J(\beta; q_2, q_1) = 2\ell n \left(\frac{q_2}{q_1} \right) \left[\frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) - 1 \right]. \quad (\text{B-23})$$

One may see that non-trivial finite and energy-dependent terms appear in the function $J(\beta)$, given by Eq.(B-21), for $q_1 = 0$ and the integration over the region $0 \leq q \leq q_2$. To show this we set $q_1 = 0$ and $q_2 = q_{\max}$ in Eq.(B-18) and arrive at the expression

$$J(\beta; q_{\max}, 0) = 2\ell n \left(\frac{q_{\max}}{\mu} \right) \left[\frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) - 1 \right] + 1 + 2\ell n 2 \left[\frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) - 1 \right] + \frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) \\ + \frac{1}{2\beta} \left[L \left(\frac{2\beta}{1+\beta} \right) - L \left(-\frac{2\beta}{1-\beta} \right) \right]. \quad (\text{B-24})$$

Due to the relation between Spence's functions [19, 21, 90]

$$L \left(\frac{2\beta}{1+\beta} \right) - L \left(-\frac{2\beta}{1-\beta} \right) = 2L \left(\frac{2\beta}{1+\beta} \right) - \frac{1}{2} \ell n^2 \left(\frac{1+\beta}{1-\beta} \right) \quad (\text{B-25})$$

we derive the following expression for $J(\beta)$

$$J(\beta; q_2, q_1) = 2\ell n \left(\frac{q_{\max}}{\mu} \right) \left[\frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) - 1 \right] + 1 + 2\ell n 2 \left[\frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) - 1 \right] + \frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) \\ - \frac{1}{4\beta} \ell n^2 \left(\frac{1+\beta}{1-\beta} \right) + \frac{1}{\beta} L \left(\frac{2\beta}{1+\beta} \right). \quad (\text{B-26})$$

In turn, using the relation [19, 21, 90]

$$L \left(\frac{2\beta}{1+\beta} \right) = L(\beta) - L(-\beta) + \frac{1}{2} \left[L \left(\frac{1-\beta}{2} \right) - L \left(\frac{1+\beta}{2} \right) \right] + \frac{1}{2} \ell n \left(\frac{1+\beta}{2} \right) \ell n \left(\frac{1+\beta}{1-\beta} \right) \quad (\text{B-27})$$

we reduce Eq.(B-26) to the form, obtained by Kinoshita and Sirlin (see Eq.(C-4) of [15]).

As a result, the functions $g_{\beta_c^- \gamma}^{(1)}(E_e, \mu)$ and $g_{\beta_c^- \gamma}^{(2)}(E_e, \mu)$, having the form

$$g_{\beta_c^- \gamma}^{(1)}(E_e, \mu) = \left[2\ell n \left(\frac{2(E_0 - E_e)}{\mu} \right) - 3 + \frac{2}{3} \frac{E_0 - E_e}{E_e} \left(1 + \frac{1}{8} \frac{E_0 - E_e}{E_e} \right) \right] \left[\frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) - 1 \right] + 1 \\ + \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2} + \frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) - \frac{1}{4\beta} \ell n^2 \left(\frac{1+\beta}{1-\beta} \right) + \frac{1}{\beta} L \left(\frac{2\beta}{1+\beta} \right), \\ g_{\beta_c^- \gamma}^{(2)}(E_e, \mu) = \left[2\ell n \left(\frac{2(E_0 - E_e)}{\mu} \right) - 3 + \frac{2}{3} \frac{E_0 - E_e}{\beta^2 E_e} \left(1 + \frac{1}{8} \frac{E_0 - E_e}{E_e} \right) \right] \left[\frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) - 1 \right] + 1 \\ + \frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) - \frac{1}{4\beta} \ell n^2 \left(\frac{1+\beta}{1-\beta} \right) + \frac{1}{\beta} L \left(\frac{2\beta}{1+\beta} \right), \quad (\text{B-28})$$

agree fully with the functions, obtained in [25] (see Eqs.(12) and (13) of Ref.[25]). We note that the same result, given by Eq.(B-26), for the logarithmically divergent integral Eq.(B-16) may be obtained within the dimensional regularisation [21]. We will use the functions Eq.(B-28) for the calculation of the radiative corrections to the rate and the correlation coefficients of the β^- -decay of the neutron (see Appendix D).

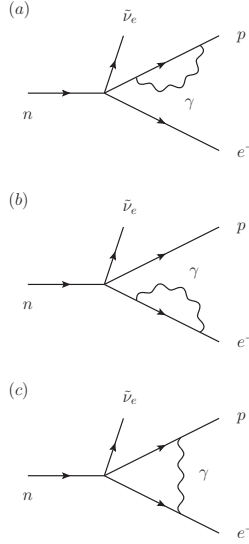


FIG. 3: Feynman diagrams of radiative corrections to the continuum-state β^- -decay of the neutron, caused by one-virtual photon exchanges.

Appendix C: Analysis of infrared divergences of radiative corrections to continuum-state β^- -decay of neutron

The radiative corrections, caused by one-virtual photon exchanges, are described by three irreducible diagrams. They are shown in Fig. 3. The diagrams in Figs. 3a and 3b define the self-energy corrections to the masses and wave functions of the proton and electron [13]–[18], respectively. As has been shown in [13]–[18], the self-energy corrections to the masses may be removed by the mass renormalisation, whereas the contributions of the self-energy diagrams to the wave functions cannot be removed fully by renormalisation of the wave functions of the proton and electron and give some observable terms. We calculate the contributions of the self-energy diagrams in Appendix D. In this Appendix we analyse the contribution of the vertex diagram in Fig. 3c.

The contribution of the vertex diagram to the amplitude of the continuum-state β^- -decay of the neutron may be written as (see Appendix D)

$$M^{(\gamma)}(n \rightarrow pe^- \bar{\nu}_e) = e^2 \frac{G_F}{\sqrt{2}} V_{ud} \mathcal{M}^{(\gamma)}. \quad (\text{C-1})$$

The amplitude $\mathcal{M}^{(\gamma)}$ is defined by

$$\mathcal{M}^{(\gamma)} = 4(k_e \cdot k_p) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}] \int \frac{d^4 q}{(2\pi)^4 i} \frac{1}{[q^2 + i0][q^2 - 2k_e \cdot q + i0][q^2 + 2k_p \cdot q + i0]} + \dots, \quad (\text{C-2})$$

where we have kept only the term, which suffers from the infrared divergences. The detailed calculation of the vertex diagram in Fig. 3c is given in Appendix D.

For the calculation of the integral over the 4-momentum q we follow the standard procedure, using Feynman's parametrisation of the integrals [13, 18] (see also [91]). We perform the calculation of the 4-momentum integral in Eq.(C-2) by using the finite-photon mass regularisation [13]–[18].

A. Finite-photon mass regularisation

Below we calculate the integral in Eq.(C-2) by using the finite-photon mass (FPM) regularisation of the infrared divergences. Applying Feynman's parametrisation [91] we transform the integral in Eq.(C-2) into the form

$$J(E_e; \mu) = \frac{1}{16\pi^2} \int_0^1 dx \int_0^1 dy 2y \int \frac{d^4 q}{\pi^2 i} \frac{1}{[(q - p(x)y)^2 - p^2(x)y^2 - \mu^2(1 - y) + i0]^3}, \quad (\text{C-3})$$

where $p(x) = k_e x - k_p(1-x)$ and μ is an infinitesimal photon mass, introduced for the infrared regularisation of the 4-momentum integrals [18, 91]. For the derivation of Eq.(C-3) we have used the relation [91]

$$\frac{1}{A_e A_p} = \int_0^1 \frac{dx}{[A_e x + A_p(1-x)]^2} \quad , \quad \frac{1}{A^2 B} = \int_0^1 \frac{2y dy}{[Ay + B(1-y)]^3}, \quad (\text{C-4})$$

where $A_e = Q^2 - 2k_e \cdot Q + i0$, $A_p = Q^2 + 2k_p \cdot Q + i0$, $A = Q^2 - 2p(x) \cdot Q + i0$ and $B = Q^2 - \mu^2 + i0$, respectively. Making a shift of variables $q - p(x)y \rightarrow q$, a Wick rotation $q_0 \rightarrow iq_4$ [91] and integrating over y we arrive at the expression [18, 19]

$$J(E_e; \mu) = -\frac{1}{32\pi^2} \int_0^1 \frac{dx}{p^2(x)} \ln \left[\frac{p^2(x)}{\mu^2} \right], \quad (\text{C-5})$$

where $p^2(x) = m_e^2 x^2 + m_p^2(1-x)^2 - 2m_e m_p \gamma x(1-x)$ with $\gamma = 1/\sqrt{1-\beta^2}$. The r.h.s. of Eq.(C-5) may be represented in the form

$$J(E_e; \mu) = -\frac{1}{32\pi^2} \frac{1}{m_e m_p c} \int_0^1 \frac{dx}{(a-x)^2 - b^2} \ln \left[\frac{m_e m_p c}{\mu^2} \left((a-x)^2 - b^2 \right) \right], \quad (\text{C-6})$$

where we have denoted

$$a = \frac{\rho + \gamma}{c}, \quad b = \frac{\sqrt{\gamma^2 - 1}}{c}, \quad c = \frac{1}{\rho} + \rho + 2\gamma \quad (\text{C-7})$$

with $\rho = m_p/m_e$. Following then [18, 19], we make a change of variables $a - x = b \coth \varphi$. This gives

$$\begin{aligned} J(E_e; \mu) &= -\frac{1}{32\pi^2} \frac{1}{m_e m_p b c} \int_{\varphi_1}^{\varphi_2} d\varphi \ln \left[\frac{m_e m_p c}{\mu^2} \frac{b^2}{\sinh^2 \varphi} \right] = \\ &= -\frac{1}{32\pi^2} \frac{1}{m_e m_p b c} \left\{ \ln \left[\frac{4m_e m_p c}{\mu^2} b^2 \right] (\varphi_2 - \varphi_1) - (\varphi_2^2 - \varphi_1^2) - 2 \int_{\varphi_1}^{\varphi_2} d\varphi \ln(1 - e^{-2\varphi}) \right\}, \end{aligned} \quad (\text{C-8})$$

where φ_2 and φ_1 are equal to

$$\begin{aligned} \varphi_2 &= \frac{1}{2} \ln \left(\frac{1-a-b}{1-a+b} \right) = \frac{1}{2} \ln \left(\frac{\frac{\rho}{1+\gamma+\sqrt{\gamma^2-1}}}{\frac{\rho}{1+\gamma-\sqrt{\gamma^2-1}}} \right), \\ \varphi_1 &= \frac{1}{2} \ln \left(\frac{a+b}{a-b} \right) = \frac{1}{2} \ln \left(\frac{\rho+\gamma+\sqrt{\gamma^2-1}}{\rho+\gamma-\sqrt{\gamma^2-1}} \right). \end{aligned} \quad (\text{C-9})$$

For the calculation of the last integral in Eq.(C-8) we make a change of variables $\varphi = -\frac{1}{2} \ln t$. This gives

$$J(\beta; \mu) = -\frac{1}{32\pi^2} \frac{1}{m_e m_p b c} \left\{ \ln \left[\frac{4m_e m_p c}{\mu^2} b^2 \right] (\varphi_2 - \varphi_1) - (\varphi_2^2 - \varphi_1^2) + L(e^{-2\varphi_2}) - L(e^{-2\varphi_1}) \right\}, \quad (\text{C-10})$$

where the last two terms are the Spence functions, defined by Eq.(B-22).

Keeping the leading order contributions in the large κ expansion we arrive at the expression

$$\begin{aligned} J(E_e; \mu) &= -\frac{1}{32\pi^2} \frac{1}{m_e m_p \sqrt{\gamma^2-1}} \left\{ -\ln \left[\frac{4m_e^2}{\mu^2} (\gamma^2-1) \right] \frac{1}{2} \ln \left(\frac{\gamma+\sqrt{\gamma^2-1}}{\gamma-\sqrt{\gamma^2-1}} \right) - \frac{1}{4} \ln^2 \left(\frac{\gamma+\sqrt{\gamma^2-1}}{\gamma-\sqrt{\gamma^2-1}} \right) \right. \\ &\quad \left. + L \left(\frac{\gamma+\sqrt{\gamma^2-1}}{\gamma-\sqrt{\gamma^2-1}} \right) - L(1) \right\}. \end{aligned} \quad (\text{C-11})$$

In terms of the electron velocity β it reads

$$J(\beta; \mu) = \frac{1}{32\pi^2} \frac{1}{E_e m_p \beta} \left\{ \ln \left[\frac{4m_e^2}{\mu^2} \frac{\beta^2}{1-\beta^2} \right] \frac{1}{2} \ln \left(\frac{1+\beta}{1-\beta} \right) + \frac{1}{4} \ln^2 \left(\frac{1+\beta}{1-\beta} \right) - L \left(\frac{1+\beta}{1-\beta} \right) + L(1) \right\}, \quad (\text{C-12})$$

where $E_e = m_e/\sqrt{1-\beta^2}$. For further transformation of the r.h.s. of Eq.(C-12) we use the following relation for the Spence functions [90]

$$\begin{aligned} L\left(\frac{1+\beta}{1-\beta}\right) - L(1) &= -L\left(\frac{1-\beta}{1+\beta}\right) + \frac{1}{2} \ln^2\left(\frac{1+\beta}{1-\beta}\right) - \frac{\pi^2}{6}, \\ L\left(\frac{1-\beta}{1+\beta}\right) &= -L\left(\frac{2\beta}{1+\beta}\right) - \ln\left(\frac{2\beta}{1-\beta}\right) \ln\left(\frac{1+\beta}{1-\beta}\right) - \frac{\pi^2}{6}, \\ L\left(\frac{1+\beta}{1-\beta}\right) - L(1) &= L\left(\frac{2\beta}{1+\beta}\right) + \ln\left(\frac{2\beta}{1-\beta}\right) \ln\left(\frac{1+\beta}{1-\beta}\right) + \frac{1}{2} \ln^2\left(\frac{1+\beta}{1-\beta}\right). \end{aligned} \quad (\text{C-13})$$

Substituting Eq.(C-13) into Eq.(C-12) we obtain

$$J(\beta; \mu) = \frac{1}{32\pi^2} \frac{1}{E_e m_p \beta} \left\{ \ln\left(\frac{m_e}{\mu}\right) \ln\left(\frac{1+\beta}{1-\beta}\right) + \frac{1}{4} \ln^2\left(\frac{1+\beta}{1-\beta}\right) - L\left(\frac{2\beta}{1+\beta}\right) \right\}. \quad (\text{C-14})$$

Thus, the integral under consideration is equal to

$$\begin{aligned} J(E_e; \mu) &= \int \frac{d^4 q}{(2\pi)^4 i} \frac{1}{[q^2 + i0][q^2 - 2k_e \cdot q + i0][q^2 + 2k_p \cdot q + i0]} \\ &= \frac{1}{32\pi^2} \frac{1}{E_e m_p \beta} \left\{ \ln\left(\frac{m_e}{\mu}\right) \ln\left(\frac{1+\beta}{1-\beta}\right) + \frac{1}{4} \ln^2\left(\frac{1+\beta}{1-\beta}\right) - L\left(\frac{2\beta}{1+\beta}\right) \right\}. \end{aligned} \quad (\text{C-15})$$

This result agrees with the expression, obtained in [18] and [19] (see Eq.(B21) of Ref.[19]).

Appendix D: Total contribution of one-virtual photon exchanges to continuum-state β^- -decay of neutron

The contributions of one-virtual photon exchanges to the continuum-state β^- -decay of the neutron are shown in Fig. 3. The correction to the amplitude of the continuum-state β^- -decay of the neutron, caused by one-virtual photon exchanges, we represent in the following form

$$M^{(\gamma)}(n \rightarrow pe^- \bar{\nu}_e) = -\frac{G_F}{\sqrt{2}} V_{ud} \left(\mathcal{M}_{pp}^{(\gamma)} + \mathcal{M}_{ee}^{(\gamma)} + \mathcal{M}_{pe}^{(\gamma)} \right), \quad (\text{D-1})$$

where the amplitudes $\mathcal{M}_{pp}^{(\gamma)}$ and $\mathcal{M}_{ee}^{(\gamma)}$ define the contributions of the self-energy diagrams of the proton and electron in Fig. 3a and Fig. 3b, respectively, and $\mathcal{M}_{pe}^{(\gamma)}$ is defined by the vertex diagram in Fig. 3c. Following [19] we calculate the diagrams in Fig. 3 in the Feynman gauge. As a result they are determined by the following analytical expressions

$$\begin{aligned} \mathcal{M}_{pp}^{(\gamma)} &= e^2 \int \frac{d^4 q}{(2\pi)^4 i} \frac{1}{q^2 + i0} \left[\bar{u}_p \gamma^\alpha \frac{1}{m_p - \hat{k}_p + \hat{q} - i0} \gamma_\alpha \frac{1}{m_p - \hat{k}_p - i0} \gamma^\mu (1 + \lambda \gamma^5) u_n \right] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}] \\ &+ [\bar{u}_p \left(-\delta m_p + \frac{Z_2^{(p)} - 1}{2} (m_p - \hat{k}_p) \right) \frac{1}{m_p - \hat{k}_p - i0} \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}], \\ \mathcal{M}_{ee}^{(\gamma)} &= e^2 [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] \int \frac{d^4 q}{(2\pi)^4 i} \frac{1}{q^2 + i0} \left[\bar{u}_e \gamma^\alpha \frac{1}{m_e - \hat{k}_e - \hat{q} - i0} \gamma_\alpha \frac{1}{m_e - \hat{k}_e - i0} \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}} \right] \\ &+ [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \left(-\delta m_e + \frac{Z_2^{(e)} - 1}{2} (m_e - \hat{k}_e) \right) \frac{1}{m_e - \hat{k}_e - i0} \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}], \\ \mathcal{M}_{pe}^{(\gamma)} &= -e^2 \int \frac{d^4 q}{(2\pi)^4 i} \frac{1}{q^2 + i0} \left[\bar{u}_p \gamma^\alpha \frac{1}{m_p - \hat{k}_p + \hat{q} - i0} \gamma^\mu (1 + \lambda \gamma^5) u_n \right] \left[\bar{u}_e \gamma_\alpha \frac{1}{m_e - \hat{k}_e - \hat{q} - i0} \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}} \right], \end{aligned} \quad (\text{D-2})$$

where $(\delta m_p, Z_2^{(p)})$ and $(\delta m_e, Z_2^{(e)})$ are the renormalisation constants of the masses and wave functions of the proton and electron, respectively.

The dependence on the electron energy is defined by the amplitude $\mathcal{M}_{pe}^{(\gamma)}$ only. For the calculation of $\mathcal{M}_{pe}^{(\gamma)}$ we reduce it to the form

$$\begin{aligned} \mathcal{M}_{pe}^{(\gamma)} &= -\frac{\alpha}{4\pi} \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} \\ &\times [\bar{u}_p \gamma^\alpha (m_p + \hat{k}_p - \hat{q}) \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\alpha (m_e + \hat{k}_e + \hat{q}) \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}], \end{aligned} \quad (\text{D-3})$$

where we have set $e^2 = 4\pi\alpha$. The numerator of the integrand of $\mathcal{M}_{pe}^{(\gamma)}$ we transcribe into the form

$$\begin{aligned}
& [\bar{u}_p \gamma^\alpha (m_p + \hat{k}_p - \hat{q}) \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\alpha (m_e + \hat{k}_e + \hat{q}) \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] = \\
& = [\bar{u}_p ((m_p - \hat{k}_p + \hat{q}) \gamma^\alpha + 2(k_p - q)_\alpha) \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e ((m_e - \hat{k}_e - \hat{q}) \gamma_\alpha + 2(k_e + q)_\alpha) \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] = \\
& = [\bar{u}_p (\hat{q} \gamma^\alpha + 2(k_p - q)_\alpha) \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e (-\hat{q} \gamma_\alpha + 2(k_e + q)_\alpha) \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] = \\
& = -[\bar{u}_p \hat{q} \gamma^\alpha \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \hat{q} \gamma_\alpha \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] + [\bar{u}_p 2\hat{q}(\hat{k}_e + \hat{q}) \gamma_\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \\
& - [\bar{u}_p \gamma_\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e 2\hat{q}(\hat{k}_p - \hat{q}) \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] + 4(k_e + q) \cdot (k_p - q) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] = \\
& = -[\bar{u}_p \hat{q} \gamma^\alpha \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \hat{q} \gamma_\alpha \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] + 2q^2 [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \\
& + (q^2 + 2k_e \cdot q) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] + (q^2 - 2k_p \cdot q) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \\
& - 2i [\bar{u}_p \sigma_{\alpha\beta} q^\alpha k_e^\beta \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] + 2i [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \sigma_{\alpha\beta} q^\alpha k_p^\beta \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \\
& - 2(q^2 + 2k_e \cdot q) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] - 2(q^2 - 2k_p \cdot q) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \\
& + 4(k_e \cdot k_p) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] = -[\bar{u}_p \hat{q} \gamma^\alpha \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \hat{q} \gamma_\alpha \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \\
& + 2q^2 [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] - 2i [\bar{u}_p \sigma_{\alpha\beta} q^\alpha k_p^\beta \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \\
& + 2i [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \sigma_{\alpha\beta} q^\alpha k_p^\beta \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] - (q^2 + 2k_e \cdot q) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \\
& - (q^2 - 2k_p \cdot q) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] + 4(k_e \cdot k_p) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}], \tag{D-4}
\end{aligned}$$

where we have used the identity $\gamma^\alpha \gamma^\beta = g^{\alpha\beta} - i\sigma^{\alpha\beta}$. For the transformation of the product

$$-[\bar{u}_p \hat{q} \gamma^\alpha \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \hat{q} \gamma_\alpha \gamma_\mu (1 - \gamma^5) v_{\bar{v}}]$$

we propose to use the relations

$$\begin{aligned}
\hat{q} \gamma^\alpha \gamma^\mu &= \gamma^\mu q^\alpha - \gamma^\alpha q^\mu + \hat{q} g^{\alpha\mu} + i\varepsilon^{\beta\alpha\mu\lambda} \gamma_\lambda \gamma^5 q_\beta, \\
\varepsilon^{\beta\alpha\mu\lambda} \varepsilon_{\rho\alpha\mu\varphi} q_\beta q^\rho &= -2q^2 g_\varphi^\lambda + 2q^\lambda q_\varphi, \\
\lambda + \gamma^5 &= -(1 + \lambda \gamma^5) + (\lambda + 1)(1 + \gamma^5)
\end{aligned} \tag{D-5}$$

This gives

$$\begin{aligned}
& -[\bar{u}_p \hat{q} \gamma^\alpha \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \hat{q} \gamma_\alpha \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] = -4q^2 [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \\
& - 2(\lambda + 1) [\bar{u}_p \hat{q} (1 + \gamma^5) u_n] [\bar{u}_e \hat{q} (1 - \gamma^5) v_{\bar{v}}] + 2(\lambda + 1) q^2 [\bar{u}_p \gamma^\mu (1 + \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}]. \tag{D-6}
\end{aligned}$$

Substituting Eq.(D-6) into Eq.(D-4) we arrive at the following expression of the numerator of the integrand of $\mathcal{M}_{pe}^{(\gamma)}$ in Eq.(D-3)

$$\begin{aligned}
& [\bar{u}_p \gamma^\alpha (m_p + \hat{k}_p - \hat{q}) \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e (m_e + \hat{k}_e + \hat{q}) \gamma_\alpha \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] = \\
& = -(q^2 + 2k_e \cdot q) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] - (q^2 - 2k_p \cdot q) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \\
& - 2q^2 [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] + 4(k_e \cdot k_p) [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \\
& - 2(\lambda + 1) [\bar{u}_p \hat{q} (1 + \gamma^5) u_n] [\bar{u}_e \hat{q} (1 - \gamma^5) v_{\bar{v}}] + 2(\lambda + 1) q^2 [\bar{u}_p \gamma^\mu (1 + \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \\
& - 2i [\bar{u}_p \sigma_{\alpha\beta} q^\alpha k_e^\beta \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] + 2i [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \sigma_{\alpha\beta} q^\alpha k_p^\beta \gamma_\mu (1 - \gamma^5) v_{\bar{v}}]. \tag{D-7}
\end{aligned}$$

For the subsequent calculations it is convenient to represent the amplitude $\mathcal{M}_{pe}^{(\gamma)}$ as a sum of two contributions

$$\mathcal{M}_{pe}^{(\gamma)} = \bar{\mathcal{M}}_{pe}^{(\gamma)} + \delta \mathcal{M}_{pe}^{(\gamma)}, \tag{D-8}$$

where $\bar{\mathcal{M}}_{pe}^{(\gamma)}$ and $\delta \mathcal{M}_{pe}^{(\gamma)}$ are given by

$$\begin{aligned}
\bar{\mathcal{M}}_{pe}^{(\gamma)} &= \frac{\alpha}{4\pi} [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{v}}] \left\{ \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} + \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} \right. \\
& \left. + 2 \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} - 4(k_e \cdot k_p) \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} \right\}. \tag{D-9}
\end{aligned}$$

and

$$\begin{aligned}
\delta \mathcal{M}_{pe}^{(\gamma)} &= \delta^{(1)} \mathcal{M}_{pe}^{(\gamma)} + \delta^{(2)} \mathcal{M}_{pe}^{(\gamma)} + \delta^{(3)} \mathcal{M}_{pe}^{(\gamma)} = \\
&= \frac{\alpha}{4\pi} \left\{ -2(\lambda+1) [\bar{u}_p \gamma^\mu (1+\gamma^5) u_n] [\bar{u}_e \gamma_\mu (1-\gamma^5) v_{\bar{v}}] \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} \right. \\
&+ 2(\lambda+1) \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} [\bar{u}_p \hat{q} (1+\gamma^5) u_n] [\bar{u}_e \hat{q} (1-\gamma^5) v_{\bar{v}}] \\
&+ 2i \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} \left([\bar{u}_p \sigma_{\alpha\beta} q^\alpha k_e^\beta \gamma^\mu (1+\lambda\gamma^5) u_n] [\bar{u}_e \gamma_\mu (1-\gamma^5) v_{\bar{v}}] \right. \\
&\left. \left. - [\bar{u}_p \gamma^\mu (1+\lambda\gamma^5) u_n] [\bar{u}_e \sigma_{\alpha\beta} q^\alpha k_p^\beta \gamma_\mu (1-\gamma^5) v_{\bar{v}}] \right) \right\}, \tag{D-10}
\end{aligned}$$

respectively. Now let us proceed to calculate $\bar{\mathcal{M}}_{pe}^\gamma$, given by Eq.(D-9). Since, the last term in Eq.(D-9) is calculated in Appendix C, so we should calculate the first three terms only. Using the Pauli-Villars regularisation for the ultra-violet divergent integrals [35]

$$\frac{1}{q^2 + i0} \rightarrow \frac{1}{q^2 + i0} - \frac{1}{q^2 - \Lambda^2 + i0} = -\frac{1}{q^2 + i0} \frac{\Lambda^2}{q^2 - \Lambda^2 + i0}, \tag{D-11}$$

where Λ is an ultraviolet cut-off, then Feynman's unification of the denominators, the shift of the virtual 4-momentum and the Wick rotation [35] for the first three integrals in Eq.(D-9) we obtain the following expressions

$$\begin{aligned}
&\int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} \rightarrow \int \frac{d^4 q}{\pi^2 i} \left[\frac{1}{q^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} - \frac{1}{q^2 - \Lambda^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} \right] = \\
&= \int_0^1 dx \int \frac{d^4 q}{\pi^2} \left[\frac{1}{(q^2 + m_p^2 x^2)^2} - \frac{1}{(q^2 + m_p^2 x^2 + \Lambda^2(1-x))^2} \right] = \int_0^1 dx \ln \left[\frac{\Lambda^2(1-x)}{m_p^2 x^2} \right] = 2\ell n \left(\frac{\Lambda}{m_p} \right) + 1, \\
&\int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} = 2\ell n \left(\frac{\Lambda}{m_e} \right) + 1, \\
&\int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} = \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{q^2}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} \rightarrow \\
&\rightarrow \int_0^1 dx \int_0^1 dy \int \frac{d^4 q}{\pi^2 i} \left[\frac{q^2}{(q - p(x)y)^2 - p^2(x)y^2 - \mu^2(1-y))^3} - \frac{q^2}{(q - p(x)y)^2 - p^2(x)y^2 - \Lambda^2(1-y))^3} \right] = \\
&= \int_0^1 dx \int_0^1 dy \int \frac{d^4 q}{\pi^2} \left[\frac{q^2 - p^2(x)y^2}{(q^2 + p^2(x)y^2 + \mu^2(1-y))^3} - \frac{q^2 - p^2(x)y^2}{(q^2 + p^2(x)y^2 + \Lambda^2(1-y))^3} \right] = \\
&= \int_0^1 dx \int_0^1 dy \int \frac{d^4 q}{\pi^2} \left\{ \frac{1}{(q^2 + p^2(x)y^2 + \mu^2(1-y))^2} - \frac{1}{(q^2 + p^2(x)y^2 + \Lambda^2(1-y))^2} \right. \\
&\left. - \frac{2p^2(x)y^2}{(q^2 + p^2(x)y^2 + \mu^2(1-y))^3} + \frac{2p^2(x)y^2 + \Lambda^2(1-y)}{(q^2 + p^2(x)y^2 + \Lambda^2(1-y))^3} \right\} = \int_0^1 dx \int_0^1 dy \int \frac{d^4 q}{\pi^2} \left\{ \ell n \left[\frac{\Lambda^2(1-y)}{p^2(x)y^2} \right] - \frac{1}{2} \right\} = \\
&= \int_0^1 dx \left\{ \ell n \left[\frac{\Lambda^2}{p^2(x)} \right] - 1 \right\} = 2\ell n \left(\frac{\Lambda}{m_p} \right) + 1. \tag{D-12}
\end{aligned}$$

For the calculation of the third integral we have replaced $p^2(x) = (k_e x - k_p(1-x))^2$ by $m_p^2(1-x)^2$.

For the calculation of the amplitudes $\mathcal{M}_{ee}^{(\gamma)}$ and $\mathcal{M}_{pp}^{(\gamma)}$ we have to perform the following standard transformations

$$\begin{aligned}
& \int \frac{d^4 q}{\pi^2 i} \bar{u} \frac{1}{q^2 + i0} \gamma^\alpha \frac{1}{\hat{k} + \hat{q} - m} \gamma_\alpha = \bar{u} \int_0^1 dx \int \frac{d^4 q}{\pi^2 i} \frac{4m - 2\hat{k} - 2\hat{q}}{[(q + kx) - m^2 x + k^2 x(1-x)]^2} = \\
& = \int_0^1 dx \bar{u} \int \frac{d^4 q}{\pi^2} \frac{4m - 2\hat{k}(1-x)}{[q^2 + m^2 x - k^2 x(1-x)]^2} = \int_0^1 dx \bar{u} \int \frac{d^4 q}{\pi^2} \frac{2m(1+x) + 2(1-x)(m - \hat{k})}{[q^2 + m^2 x - 2k^2 x(1-x)]^2} = \\
& = \int_0^1 dx \bar{u} \left[\int \frac{d^4 q}{\pi^2} \frac{2m(1+x)}{[q^2 + m^2 x - k^2 x(1-x)]^2} + \frac{2(1-x)}{[q^2 + m^2 x - k^2 x(1-x)]^2} (m - \hat{k}) \right. \\
& \quad \left. - \frac{4mx(1-x^2)(m + \hat{k})}{[q^2 + m_e^2 x - k^2 x(1-x)]^3} (m - \hat{k}) \right] = \bar{u} \left\{ \int_0^1 dx \int \frac{d^4 q}{\pi^2} \frac{2m(1+x)}{(q^2 + m^2 x^2)^2} + (m - \hat{k}) \int_0^1 dx \int \frac{d^4 q}{\pi^2} \left[\frac{2(1-x)}{(q^2 + m^2 x^2)^2} \right. \right. \\
& \quad \left. \left. - \frac{8m^2 x(1-x^2)}{(q^2 + m^2 x^2)^3} \right] \right\}. \tag{D-13}
\end{aligned}$$

This results in the relation

$$\begin{aligned}
\int \frac{d^4 q}{\pi^2 i} \bar{u} \frac{1}{q^2 + i0} \gamma^\alpha \frac{1}{\hat{q} + \hat{k} - m} \gamma_\alpha &= \bar{u} \left\{ \int_0^1 dx \int \frac{d^4 q}{\pi^2} \frac{2m(1+x)}{(q^2 + m^2 x^2)^2} \right. \\
&\quad \left. + (m - \hat{k}) \int_0^1 dx \int \frac{d^4 q}{\pi^2} \left[\frac{2(1-x)}{(q^2 + m^2 x^2)^2} - \frac{8m^2 x(1-x^2)}{(q^2 + m^2 x^2)^3} \right] \right\}, \tag{D-14}
\end{aligned}$$

where $m = m_e$ and $m = m_p$ for the electron and proton self-energy contributions, respectively. The term, which is not proportional to $(m - \hat{k})$, can be removed by the mass renormalisation. So the amplitudes $\mathcal{M}_{ee}^{(\gamma)}$ and $\mathcal{M}_{pp}^{(\gamma)}$ are defined by the contributions of the terms, proportional to $(m_e - \hat{k}_e)$ and $(m_p - \hat{k}_p)$, respectively. This gives

$$\begin{aligned}
\mathcal{M}_{ee}^{(\gamma)} &= [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{e}}] \left(\frac{\alpha}{4\pi} I(m_e) + \frac{Z_2^{(e)} - 1}{2} \right), \\
\mathcal{M}_{pp}^{(\gamma)} &= [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{e}}] \left(\frac{\alpha}{4\pi} I(m_p) + \frac{Z_2^{(p)} - 1}{2} \right), \tag{D-15}
\end{aligned}$$

where $I(m)$ is given by

$$I(m) = \int_0^1 dx \int \frac{d^4 q}{\pi^2} \left[-\frac{2(1-x)}{(q^2 + m^2 x^2)^2} + \frac{8m^2 x(1-x^2)}{(q^2 + m^2 x^2 + \mu^2(1-x))^3} \right] \tag{D-16}$$

for $m = m_e$ and m_p , respectively. For the ultra-violet and infrared regularisation we use the following expressions for the photon Green function

$$\frac{1}{q^2 + i0} \rightarrow \frac{1}{q^2 - \mu^2 + i0} - \frac{1}{q^2 - \Lambda^2 + i0}. \tag{D-17}$$

The regularised function $I(m)$ takes the form

$$\begin{aligned}
I(m) &= \int_0^1 dx \int \frac{d^4 q}{\pi^2} \left[-\frac{2(1-x)}{(q^2 + m^2 x^2)^2} + \frac{2(1-x)}{(q^2 + m^2 x^2 + \Lambda^2(1-x))^2} \right. \\
&\quad \left. + \frac{8m^2 x(1-x^2)}{(q^2 + m^2 x^2 + \mu^2(1-x))^3} - \frac{8m^2 x(1-x^2)}{(q^2 + m^2 x^2 + \Lambda^2(1-x))^3} \right]. \tag{D-18}
\end{aligned}$$

The calculation of the integrals in Eq.(D-18) runs as follows. For the first two integrals, which are ultra-violet divergent, we obtain the expression

$$\int_0^1 dx \int \frac{d^4 q}{\pi^2} \left[\frac{2(1-x)}{(q^2 + m^2 x^2 + \Lambda^2(1-x))^2} - \frac{2(1-x)}{(q^2 + m^2 x^2)^2} \right] = - \int_0^1 dx 2(1-x) \ln \left[\frac{(1-x)\Lambda^2}{x^2 m^2} \right] - 2 \ln \left(\frac{\Lambda}{m} \right) - \frac{5}{2}. \tag{D-19}$$

The third and fourth integrals in Eq.(D-18) are ultra-violet convergent. Moreover in the limit $\Lambda \rightarrow \infty$ the fourth integral vanishes. The third integral is infrared divergent and its calculation gives

$$\int_0^1 dx \int \frac{d^4 q}{\pi^2} \frac{8m^2 x(1-x^2)}{(q^2 + \mu^2(1-x) + m^2 x^2)^3} = 4m^2 \int_0^1 dx \frac{x(1-x^2)}{\mu^2(1-x) + m^2 x^2} = -4 \ln \left(\frac{\mu}{m} \right) - 2. \tag{D-20}$$

Thus, the function $I(m)$ takes the form

$$I(m) = -2\ell n\left(\frac{\Lambda}{m}\right) - 4\ell n\left(\frac{\mu}{m}\right) - \frac{9}{2}. \quad (\text{D-21})$$

Summing up the contributions of the amplitudes $\mathcal{M}_{ee}^{(\gamma)}$, $\mathcal{M}_{pp}^{(\gamma)}$ and $\bar{\mathcal{M}}_{pe}^{(\gamma)}$ we obtain the following expression

$$\begin{aligned} \mathcal{M}_{ee}^{(\gamma)} + \mathcal{M}_{pp}^{(\gamma)} + \bar{\mathcal{M}}_{pe}^{(\gamma)} &= [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}] \left\{ \frac{Z_2^{(e)} - 1}{2} + \frac{Z_2^{(p)} - 1}{2} \right. \\ &+ \frac{\alpha}{4\pi} \left[-2\ell n\left(\frac{\Lambda}{m_e}\right) - 4\ell n\left(\frac{\mu}{m_e}\right) - \frac{9}{2} - 2\ell n\left(\frac{\Lambda}{m_p}\right) - 4\ell n\left(\frac{\mu}{m_p}\right) - \frac{9}{2} + 2\ell n\left(\frac{\Lambda}{m_p}\right) + 1 \right. \\ &\left. \left. + 2\ell n\left(\frac{\Lambda}{m_e}\right) + 1 + 4\ell n\left(\frac{\Lambda}{m_p}\right) + 2 + 2\left\{ \ell n\left(\frac{\mu}{m_e}\right) \frac{1}{\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right) - \frac{1}{4\beta} \ell n^2\left(\frac{1+\beta}{1-\beta}\right) + \frac{1}{\beta} L\left(\frac{2\beta}{1+\beta}\right) \right\} \right] \right\}. \end{aligned} \quad (\text{D-22})$$

After the cancellation of some terms we arrive at the expression

$$\begin{aligned} \mathcal{M}_{ee}^{(\gamma)} + \mathcal{M}_{pp}^{(\gamma)} + \bar{\mathcal{M}}_{pe}^{(\gamma)} &= [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}] \left\{ \frac{Z_2^{(e)} - 1}{2} + \frac{Z_2^{(p)} - 1}{2} \right. \\ &+ \frac{\alpha}{2\pi} \left[2\ell n\left(\frac{\Lambda}{m_p}\right) - 2\ell n\left(\frac{\mu}{m_e}\right) - 2\ell n\left(\frac{\mu}{m_p}\right) - \frac{5}{2} + \frac{1}{\beta} \ell n\left(\frac{\mu}{m_e}\right) \ell n\left(\frac{1+\beta}{1-\beta}\right) - \frac{1}{4\beta} \ell n^2\left(\frac{1+\beta}{1-\beta}\right) + \frac{1}{\beta} L\left(\frac{2\beta}{1+\beta}\right) \right] \right\}. \end{aligned} \quad (\text{D-23})$$

Then, we transcribe the r.h.s. of Eq.(D-23) into the following symmetric form

$$\begin{aligned} \mathcal{M}_{ee}^{(\gamma)} + \mathcal{M}_{pp}^{(\gamma)} + \bar{\mathcal{M}}_{pe}^{(\gamma)} &= [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}] \\ &\times \left\{ \left[\frac{Z_2^{(e)} - 1}{2} - \frac{\alpha}{2\pi} \left(\frac{1}{2} \ell n\left(\frac{\Lambda}{m_e}\right) + \ell n\left(\frac{\mu}{m_e}\right) + \frac{9}{8} \right) \right] + \left[\frac{Z_2^{(p)} - 1}{2} - \frac{\alpha}{2\pi} \left(\frac{1}{2} \ell n\left(\frac{\Lambda}{m_p}\right) + \ell n\left(\frac{\mu}{m_p}\right) + \frac{9}{8} \right) \right] \right. \\ &\left. + \frac{\alpha}{2\pi} \left[3\ell n\left(\frac{\Lambda}{m_p}\right) + \frac{3}{2} \ell n\left(\frac{m_p}{m_e}\right) - \frac{1}{4} + 2\ell n\left(\frac{\mu}{m_e}\right) \left[\frac{1}{2\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right) - 1 \right] - \frac{1}{4\beta} \ell n^2\left(\frac{1+\beta}{1-\beta}\right) + \frac{1}{\beta} L\left(\frac{2\beta}{1+\beta}\right) \right] \right\}. \end{aligned} \quad (\text{D-24})$$

Defining the constants of the renormalisation of the wave functions of the electron and proton in the standard form as [35]

$$\begin{aligned} \frac{Z_2^{(e)} - 1}{2} &= \frac{\alpha}{2\pi} \left[\frac{1}{2} \ell n\left(\frac{\Lambda}{m_e}\right) + \ell n\left(\frac{\mu}{m_e}\right) + \frac{9}{8} \right], \\ \frac{Z_2^{(p)} - 1}{2} &= \frac{\alpha}{2\pi} \left[\frac{1}{2} \ell n\left(\frac{\Lambda}{m_p}\right) + \ell n\left(\frac{\mu}{m_p}\right) + \frac{9}{8} \right] \end{aligned} \quad (\text{D-25})$$

we reduce the r.h.s. of Eq.(D-24) to the form

$$\begin{aligned} \mathcal{M}_{ee}^{(\gamma)} + \mathcal{M}_{pp}^{(\gamma)} + \bar{\mathcal{M}}_{pe}^{(\gamma)} &= [\bar{u}_p \gamma^\mu (1 + \lambda \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}] \\ &\times \frac{\alpha}{2\pi} \left\{ 3\ell n\left(\frac{\Lambda}{m_p}\right) + \frac{3}{2} \ell n\left(\frac{m_p}{m_e}\right) - \frac{1}{4} + 2\ell n\left(\frac{\mu}{m_e}\right) \left[\frac{1}{2\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right) - 1 \right] + \frac{1}{\beta} L\left(\frac{2\beta}{1+\beta}\right) - \frac{1}{4\beta} \ell n^2\left(\frac{1+\beta}{1-\beta}\right) \right\}. \end{aligned} \quad (\text{D-26})$$

In the non-relativistic approximation for the proton Eq.(D-26) reads

$$\begin{aligned} \mathcal{M}_{ee}^{(\gamma)} + \mathcal{M}_{pp}^{(\gamma)} + \bar{\mathcal{M}}_{pe}^{(\gamma)} &= \frac{\alpha}{2\pi} \left\{ 3\ell n\left(\frac{\Lambda}{m_p}\right) + \frac{3}{2} \ell n\left(\frac{m_p}{m_e}\right) - \frac{1}{4} + 2\ell n\left(\frac{\mu}{m_e}\right) \left[\frac{1}{2\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right) - 1 \right] \right. \\ &\left. + \frac{1}{\beta} L\left(\frac{2\beta}{1+\beta}\right) - \frac{1}{4\beta} \ell n^2\left(\frac{1+\beta}{1-\beta}\right) \right\} \left\{ [\varphi_p^\dagger \varphi_n] [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - \lambda [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] \right\}. \end{aligned} \quad (\text{D-27})$$

Now let us take into account the contribution of the first two terms in $\delta\mathcal{M}_{pe}^{(\gamma)}$, given by Eq.(D-10). The first term in Eq.(D-10) is equal to

$$\begin{aligned} \delta^{(1)} \mathcal{M}_{pe}^{(\gamma)} &= -(\lambda + 1) \frac{\alpha}{2\pi} [\bar{u}_p \gamma^\mu (1 + \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}] \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} = \\ &= -(\lambda + 1) \frac{\alpha}{2\pi} \left[2\ell n\left(\frac{\Lambda}{m_p}\right) + 1 \right] [\bar{u}_p \gamma^\mu (1 + \gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}], \end{aligned} \quad (\text{D-28})$$

where we have used Eq.(D-12). In the non—relativistic approximation for the proton it reads

$$\delta^{(1)}\mathcal{M}_{pe}^{(\gamma)} = -2m_n(\lambda+1)\frac{\alpha}{2\pi}\left[2\ell n\left(\frac{\Lambda}{m_p}\right)+1\right]\left\{[\varphi_p^\dagger\varphi_n][\bar{u}_e\gamma^0(1-\gamma^5)v_{\bar{\nu}}] - [\varphi_p^\dagger\vec{\sigma}\varphi_n]\cdot[\bar{u}_e\vec{\gamma}(1-\gamma^5)v_{\bar{\nu}}]\right\}. \quad (\text{D-29})$$

The calculation of the second term in Eq.(D-10) runs as follows

$$\begin{aligned} \delta^{(2)}\mathcal{M}_{pe}^{(\gamma)} &= (\lambda+1)\frac{\alpha}{2\pi}[\bar{u}_p\gamma^\alpha(1+\gamma^5)u_n][\bar{u}_e\gamma^\beta(1-\gamma^5)v_{\bar{\nu}}]\int\frac{d^4q}{\pi^2i}\frac{q_\alpha q_\beta}{q^2+i0}\frac{1}{q^2-2k_p\cdot q+i0}\frac{1}{q^2+2k_e\cdot q+i0}\rightarrow \\ &\rightarrow (\lambda+1)\frac{\alpha}{2\pi}[\bar{u}_p\gamma^\alpha(1+\gamma^5)u_n][\bar{u}_e\gamma^\beta(1-\gamma^5)v_{\bar{\nu}}]\int\frac{d^4q}{\pi^2i}\left[\frac{q_\alpha q_\beta}{q^2+i0}-\frac{q_\alpha q_\beta}{q^2-\Lambda^2+i0}\right]\frac{1}{q^2-2k_p\cdot q+i0}\frac{1}{q^2+2k_e\cdot q+i0}= \\ &= (\lambda+1)\frac{\alpha}{2\pi}[\bar{u}_p\gamma^\alpha(1+\gamma^5)u_n][\bar{u}_e\gamma^\beta(1-\gamma^5)v_{\bar{\nu}}]\int_0^1dx\int_0^1dy2y\int\frac{d^4q}{\pi^2i}\left[\frac{q_\alpha q_\beta}{(q+p(x)y)^2-p^2(x)y^2)^3}\right. \\ &\quad \left.-\frac{q_\alpha q_\beta}{(q+p(x)y)^2-p^2(x)y^2-\Lambda^2(1-y)^3}\right]. \end{aligned} \quad (\text{D-30})$$

After the shift of variables, the integration over the 4-dimensional solid angle and the Wick rotation we arrive at the expression

$$\begin{aligned} \delta^{(2)}\mathcal{M}_{pe}^{(\gamma)} &= (\lambda+1)\frac{\alpha}{8\pi}[\bar{u}_p\gamma^\alpha(1+\gamma^5)u_n][\bar{u}_e\gamma^\beta(1-\gamma^5)v_{\bar{\nu}}]\int_0^1dx\int_0^1dy2y\int\frac{d^4q}{\pi^2}\left[\frac{q^2g_{\alpha\beta}-4y^2p_\alpha(x)p_\beta(x)}{(q^2+p^2(x)y^2)^3}\right. \\ &\quad \left.-\frac{q^2g_{\alpha\beta}-4y^2p_\alpha(x)p_\beta(x)}{(q^2+p^2(x)y^2+\Lambda^2(1-y))^3}\right] = (\lambda+1)\frac{\alpha}{8\pi}[\bar{u}_p\gamma^\alpha(1+\gamma^5)u_n][\bar{u}_e\gamma^\beta(1-\gamma^5)v_{\bar{\nu}}]\left\{\int_0^1dx\int_0^1dy2y\int\frac{d^4q}{\pi^2}\frac{g_{\alpha\beta}}{(q^2+p^2(x)y^2)^2}\right. \\ &\quad \left.-\int_0^1dx\int_0^1dy2y^3\int\frac{d^4q}{\pi^2}\frac{p^2(x)g_{\alpha\beta}+4p_\alpha(x)p_\beta(x)}{(q^2+p^2(x)y^2)^3}-\int_0^1dx\int_0^1dy2y\int\frac{d^4q}{\pi^2}\frac{g_{\alpha\beta}}{(q^2+p^2(x)y^2+\Lambda^2(1-y))^2}\right. \\ &\quad \left.+\int_0^1dx\int_0^1dy2y\int\frac{d^4q}{\pi^2}\frac{(p^2(x)y^2+\Lambda^2(1-y))g_{\alpha\beta}}{(q^2+p^2(x)y^2+\Lambda^2(1-y))^3}+\int_0^1dx\int_0^1dy2y\int\frac{d^4q}{\pi^2}\frac{4y^2p_\alpha(x)p_\beta(x)}{(q^2+p^2(x)y^2+\Lambda^2(1-y))^3}\right\}. \end{aligned} \quad (\text{D-31})$$

After the integration over q the term $\delta^{(2)}\mathcal{M}_{pe}^{(\gamma)}$ is

$$\delta^{(2)}\mathcal{M}_{pe}^{(\gamma)} = (\lambda+1)\frac{\alpha}{8\pi}[\bar{u}_p\gamma^\alpha(1+\gamma^5)u_n][\bar{u}_e\gamma^\beta(1-\gamma^5)v_{\bar{\nu}}]\left\{g_{\alpha\beta}\int_0^1dx\int_0^1dy2y\ell n\left[\frac{\Lambda^2(1-y)}{p^2(x)y^2}\right]-2\int_0^1dx\frac{p_\alpha(x)p_\beta(x)}{p^2(x)}\right\}. \quad (\text{D-32})$$

Keeping the leading terms in the large m_p expansion, i.e. replacing $p^2(x) = (k_ex - k_p(1-x))^2$ by $p^2(x) \rightarrow m_p^2(1-x)^2$, and integrating over y and x we obtain

$$\delta^{(2)}\mathcal{M}_{pe}^{(\gamma)} = (\lambda+1)\frac{\alpha}{4\pi}[\bar{u}_p\gamma^\alpha(1+\gamma^5)u_n][\bar{u}_e\gamma^\beta(1-\gamma^5)v_{\bar{\nu}}]\left\{g_{\alpha\beta}\left[\ell n\left(\frac{\Lambda}{m_p}\right)+\frac{3}{2}\right]-g_{\alpha\alpha}g_{\beta\beta}\right\}. \quad (\text{D-33})$$

In the non—relativistic approximation for the proton Eq.(D-33) takes the form

$$\begin{aligned} \delta^{(2)}\mathcal{M}_{pe}^{(\gamma)} &= 2m_n\frac{\alpha}{2\pi}\left\{(\lambda+1)\left[\frac{1}{2}\ell n\left(\frac{\Lambda}{m_p}\right)+\frac{1}{4}\right]\right\}[\varphi_p^\dagger\varphi_n][\bar{u}_e\gamma^0(1-\gamma^5)v_{\bar{\nu}}] \\ &\quad + 2m_n\frac{\alpha}{2\pi}\left\{(\lambda+1)\left[-\frac{1}{2}\ell n\left(\frac{\Lambda}{m_p}\right)-\frac{3}{4}\right]\right\}[\varphi_p^\dagger\vec{\sigma}\varphi_n]\cdot[\bar{u}_e\vec{\gamma}(1-\gamma^5)v_{\bar{\nu}}]. \end{aligned} \quad (\text{D-34})$$

The sum of the contributions of $\delta^{(1)}\mathcal{M}_{pe}^{(\gamma)}$ and $\delta^{(2)}\mathcal{M}_{pe}^{(\gamma)}$ is

$$\begin{aligned} \delta^{(1)}\mathcal{M}_{pe}^{(\gamma)} + \delta^{(2)}\mathcal{M}_{pe}^{(\gamma)} &= 2m_n\left\{\frac{\alpha}{2\pi}(\lambda+1)\left[-\frac{3}{2}\ell n\left(\frac{\Lambda}{m_p}\right)-\frac{3}{4}\right]\right\}[\varphi_p^\dagger\varphi_n][\bar{u}_e\gamma^0(1-\gamma^5)v_{\bar{\nu}}] \\ &\quad + 2m_n\left\{\frac{\alpha}{2\pi}(\lambda+1)\left[+\frac{3}{2}\ell n\left(\frac{\Lambda}{m_p}\right)+\frac{1}{4}\right]\right\}[\varphi_p^\dagger\vec{\sigma}\varphi_n]\cdot[\bar{u}_e\vec{\gamma}(1-\gamma^5)v_{\bar{\nu}}]. \end{aligned} \quad (\text{D-35})$$

Now we have to calculate the contribution of the last term $\delta^{(3)}\mathcal{M}_{pe}^{(\gamma)}$ to the amplitude $\delta\mathcal{M}_{pe}^{(\gamma)}$, given by

$$\begin{aligned} \delta^{(3)}\mathcal{M}_{pe}^{(\gamma)} &= \frac{\alpha}{2\pi}\int\frac{d^4q}{\pi^2i}\frac{1}{q^2+i0}\frac{1}{q^2-2k_p\cdot q+i0}\frac{1}{q^2+2k_e\cdot q+i0} \\ &\quad \times\left([\bar{u}_p i\sigma_{\alpha\beta}q^\alpha k_e^\beta\gamma^\mu(1+\lambda\gamma^5)u_n][\bar{u}_e\gamma_\mu(1-\gamma^5)v_{\bar{\nu}}] - [\bar{u}_p\gamma^\mu(1+\lambda\gamma^5)u_n][\bar{u}_e i\sigma_{\alpha\beta}q^\alpha k_p^\beta\gamma_\mu(1-\gamma^5)v_{\bar{\nu}}]\right). \end{aligned} \quad (\text{D-36})$$

Following the procedure expounded above, we reduce the r.h.s of Eq.(D-36) to the form

$$\begin{aligned} \delta^{(3)}\mathcal{M}_{pe}^{(\gamma)} = & \frac{\alpha}{2\pi} \left\{ [\bar{u}_p i\sigma_{\alpha\beta} k_e^\alpha k_p^\beta \gamma^\mu (1 + \lambda\gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}] \int_0^1 \frac{dx (1-x)}{p^2(x)} \right. \\ & \left. - [\bar{u}_p \gamma^\mu (1 + \lambda\gamma^5) u_n] [\bar{u}_e i\sigma_{\alpha\beta} k_e^\alpha k_p^\beta \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}] \int_0^1 \frac{dx x}{p^2(x)} \right\}, \end{aligned} \quad (\text{D-37})$$

where $p^2(x) = (k_e x - k_p(1-x))^2 = m_e^2 x^2 + m_p^2(1-x)^2 - 2m_e m_p \gamma x(1-x)$.

The calculation of the integrals over x in Eq.(D-37) has been carried out in detail in Appendix C. Using these results we obtain

$$\begin{aligned} \int_0^1 \frac{dx}{p^2(x)} &= -\frac{1}{m_e m_p} \frac{\sqrt{1-\beta^2}}{2\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right), \\ \int_0^1 \frac{dx x}{p^2(x)} &= -\frac{1}{m_e m_p} \frac{\sqrt{1-\beta^2}}{2\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right) \\ &\quad - \frac{1}{m_p^2} \left[\ell n\left(\frac{m_p}{m_e}\right) - \frac{1}{2\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right) \right] + \dots, \\ \int_0^1 \frac{dx (1-x)}{p^2(x)} &= \frac{1}{m_p^2} \left[\ell n\left(\frac{m_p}{m_e}\right) - \frac{1}{2\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right) \right] + \dots, \end{aligned} \quad (\text{D-38})$$

where the ellipses denote the contributions of the terms of higher order in the large m_p expansion.

Keeping the leading terms in the large m_p expansion, the contribution of $\delta^{(3)}\mathcal{M}_{pe}^{(\gamma)}$ is

$$\delta^{(3)}\mathcal{M}_{pe}^{(\gamma)} = \frac{\alpha}{2\pi} \left[\frac{\sqrt{1-\beta^2}}{2\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right) \right] [\bar{u}_p \gamma^\mu (1 + \lambda\gamma^5) u_n] [\bar{u}_e \frac{E_e - m_e \gamma^0}{m_e} \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}], \quad (\text{D-39})$$

where we have used the Dirac equation for the free electron $\bar{u}_e(\vec{k}_e \cdot \vec{\gamma}) = \bar{u}_e(E_e \gamma^0 - m_e)$. Thus $\delta^{(3)}\mathcal{M}_{pe}^{(\gamma)}$ is given by

$$\begin{aligned} \delta^{(3)}\mathcal{M}_{pe}^{(\gamma)} &= \frac{\alpha}{2\pi} \left[\frac{1}{2\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right) \right] [\bar{u}_p \gamma^\mu (1 + \lambda\gamma^5) u_n] [\bar{u}_e \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}] \\ &\quad - \frac{\alpha}{2\pi} \left[\frac{\sqrt{1-\beta^2}}{2\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right) \right] [\bar{u}_p \gamma^\mu (1 + \lambda\gamma^5) u_n] [\bar{u}_e \gamma^0 \gamma_\mu (1 - \gamma^5) v_{\bar{\nu}}]. \end{aligned} \quad (\text{D-40})$$

In the non-relativistic approximation for the proton Eq.(D-40) reads

$$\begin{aligned} \delta^{(3)}\mathcal{M}_{pe}^{(\gamma)} &= 2m_n \frac{\alpha}{2\pi} \frac{1}{2\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right) \left\{ [\varphi_p^\dagger \varphi_n] [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - \lambda [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] \right\} \\ &\quad - 2m_n \frac{\alpha}{2\pi} \frac{\sqrt{1-\beta^2}}{2\beta} \ell n\left(\frac{1+\beta}{1-\beta}\right) \left\{ [\varphi_p^\dagger \varphi_n] [\bar{u}_e (1 - \gamma^5) v_{\bar{\nu}}] - \lambda [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \gamma^0 \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] \right\}. \end{aligned} \quad (\text{D-41})$$

The second term can be identified with the contributions of the scalar and tensor lepton-nucleon weak interactions. In order to show this we use the Hamilton density operator of weak interactions, taken in the following form [10]

$$\begin{aligned} \mathcal{H}_W(x) &= \frac{G_F}{\sqrt{2}} V_{ud} \left\{ [\bar{\psi}_p(x) \gamma_\mu (1 + \lambda\gamma^5) \psi_n(x)] [\bar{\psi}_e(x) \gamma^\mu (1 - \gamma^5) \psi_{\nu_e}(x)] \right. \\ &\quad \left. + g_S [\bar{\psi}_p(x) \psi_n(x)] [\bar{\psi}_e(x) (1 - \gamma^5) \psi_{\nu_e}(x)] + \frac{1}{2} g_T [\bar{\psi}_p(x) \sigma_{\mu\nu} \gamma^5 \psi_n(x)] [\bar{\psi}_e(x) \sigma^{\mu\nu} (1 - \gamma^5) \psi_{\nu_e}(x)] \right\}, \end{aligned} \quad (\text{D-42})$$

where g_S and g_T are the constants of scalar and tensor weak interactions and $\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$ is the Dirac matrix.

In the rest frame of the neutron and in the non-relativistic approximation for the proton the amplitude of the continuum-state β^- -decay of the neutron takes the form [10]

$$M(n \rightarrow p + e^- + \bar{\nu}_e) = -2m_n \frac{G_F}{\sqrt{2}} V_{ud} \left\{ [\varphi_p^\dagger \varphi_n] [\bar{u}_e (1 + g_S \gamma^0) \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] + [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e (-\lambda + g_T \gamma^0) \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] \right\}. \quad (\text{D-43})$$

From the comparison Eq.(D-41) with Eq.(D-43) we define the scalar and tensor coupling constants as

$$\begin{aligned} g_S(E_e) &= -\frac{\alpha}{2\pi} \frac{\sqrt{1-\beta^2}}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) = -\frac{\alpha}{2\pi} g_F(E_e), \\ g_T(E_e) &= +\frac{\alpha}{2\pi} \lambda \frac{\sqrt{1-\beta^2}}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) = +\frac{\alpha}{2\pi} \lambda g_F(E_e). \end{aligned} \quad (\text{D-44})$$

Using the results, obtained in [10], we may define the contribution of the electromagnetic Fierz term [10]

$$b_F^{(\text{em})}(E_e) = 2 \frac{g_S(E_e) - 3\lambda g_T(E_e)}{1 + 3\lambda^2} = -\frac{\alpha}{\pi} g_F(E_e), \quad (\text{D-45})$$

induced by one-virtual photon exchanges. Summing up the contributions of $\delta^{(j)}\mathcal{M}_{pe}$ with $j = 1, 2, 3$ we obtain the following expression for $\delta\mathcal{M}_{pe}^{(\gamma)}$

$$\begin{aligned} \delta\mathcal{M}_{pe}^{(\gamma)} &= \delta^{(1)}\mathcal{M}_{pe}^{(\gamma)} + \delta^{(2)}\mathcal{M}_{pe}^{(\gamma)} + \delta^{(3)}\mathcal{M}_{pe}^{(\gamma)} = 2m_n \frac{\alpha}{2\pi} \left\{ (\lambda + 1) \left[-\frac{3}{2} \ln\left(\frac{\Lambda}{m_p}\right) - \frac{3}{4} \right] + \frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) \right\} \\ &\times [\varphi_p^\dagger \varphi_n] [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] + 2m_n \frac{\alpha}{2\pi} \left\{ (\lambda + 1) \left[+\frac{3}{2} \ln\left(\frac{\Lambda}{m_p}\right) + \frac{1}{4} \right] - \lambda \frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) \right\} [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] \\ &- 2m_n \frac{\alpha}{2\pi} g_F(E_e) \left\{ [\varphi_p^\dagger \varphi_n] [\bar{u}_e (1 - \gamma^5) v_{\bar{\nu}}] - \lambda [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \gamma^0 \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] \right\}. \end{aligned} \quad (\text{D-46})$$

Summing up the contributions of Eq.(D-27) and Eq.(D-46) we calculate $\mathcal{M}_{pp}^{(\gamma)} + \mathcal{M}_{ee}^{(\gamma)} + \mathcal{M}_{pe}^{(\gamma)}$. Using this expression we get the amplitude of the continuum-state β^- -decay of the neutron, calculated in the non-relativistic approximation for the proton and taking into account the contributions of one-virtual photon exchanges. We represent it in the form

$$\begin{aligned} M(n \rightarrow pe^- \bar{\nu}_e) &= -2m_n \frac{G_F^{(r)}}{\sqrt{2}} V_{ud} \left\{ \left(1 + \frac{\alpha}{2\pi} f_{\beta_c^-}(E_e, \mu) \right) [\varphi_p^\dagger \varphi_n] [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] \right. \\ &- \lambda^{(r)} \left(1 + \frac{\alpha}{2\pi} f_{\beta_c^-}(E_e, \mu) \right) [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] - \frac{\alpha}{2\pi} g_F(E_e) [\varphi_p^\dagger \varphi_n] [\bar{u}_e (1 - \gamma^5) v_{\bar{\nu}}] \\ &\left. + \frac{\alpha}{2\pi} \lambda^{(r)} g_F(E_e) [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \gamma^0 \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] \right\}. \end{aligned} \quad (\text{D-47})$$

Here $G_F^{(r)}$ and $g_A^{(r)}$ are the renormalised Fermi and axial coupling constants

$$G_F^{(r)} = G_F \left(1 + \frac{\alpha}{2\pi} d_V \right), \quad \lambda^{(r)} = \lambda \left(1 + \frac{\alpha}{2\pi} d_A \right), \quad (\text{D-48})$$

where d_V and d_A are ultra-violet divergent constants. In our calculation of the radiative corrections they are equal to

$$\begin{aligned} d_V &= 3 \ln\left(\frac{\Lambda}{m_p}\right) - \frac{1}{4} - c_S - (\lambda + 1) \left[\frac{3}{2} \ln\left(\frac{\Lambda}{m_p}\right) + \frac{3}{4} \right], \\ d_A &= -\frac{\lambda + 1}{\lambda} \left[\frac{3}{2} \ln\left(\frac{\Lambda}{m_p}\right) + \frac{1}{4} \right] + (\lambda + 1) \left[\frac{3}{2} \ln\left(\frac{\Lambda}{m_p}\right) + \frac{3}{4} \right]. \end{aligned} \quad (\text{D-49})$$

The function $f_{\beta_c^-}(E_e, \mu)$, given by

$$f_{\beta_c^-}(E_e, \mu) = \frac{3}{2} \ln\left(\frac{m_p}{m_e}\right) + c_S + 2 \ln\left(\frac{\mu}{m_e}\right) \left[\frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 1 \right] + \frac{1}{\beta} L\left(\frac{2\beta}{1+\beta}\right) - \frac{1}{4\beta} \ln^2\left(\frac{1+\beta}{1-\beta}\right) + \frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right), \quad (\text{D-50})$$

contains two terms $(3/2)\ln(m_p/m_e)$ and c_S independent of the electron energy E_e . As we show in Appendix E the term $(3/2)\ln(m_p/m_e)$ is fixed by the KLN theorem. In our calculation the constant c_S reflects an ambiguous decomposition of the contribution of one-virtual photon exchanges to the amplitude of the neutron β^- -decay into the renormalisation constant d_V of the Fermi coupling constant and the radiative corrections to the lifetime of the neutron. Nevertheless, following Sirlin [18] and Abers *et al.* [19] one can show that due to a requirement of gauge invariance of the observable radiative corrections the value of the constant c_S is fixed and is equal to $c_S = -11/8$. We show this in Appendix F.

The contribution of the W -boson and Z -boson exchanges and the QCD corrections [28] we describe by the constant C_{WZ} [32]. The numerical value $C_{WZ} = 10.249$ we discuss below Eq.(D-58). As a result the function $f_{\beta_c^-}(E_e, \mu)$ is

$$f_{\beta_c^-}(E_e, \mu) = \frac{3}{2} \ln\left(\frac{m_p}{m_e}\right) - \frac{11}{8} + 2 \ln\left(\frac{\mu}{m_e}\right) \left[\frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 1 \right] + \frac{1}{\beta} L\left(\frac{2\beta}{1+\beta}\right) - \frac{1}{4\beta} \ln^2\left(\frac{1+\beta}{1-\beta}\right) + \frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) + C_{WZ}. \quad (\text{D-51})$$

Making a replacement $G_F^{(r)} \rightarrow G_F$ and $\lambda^{(r)} \rightarrow \lambda$ and taking into account the contributions of the “weak magnetism” and the proton recoil the amplitude of the continuum-state β^- -decay of the neutron takes the form

$$\begin{aligned} M(n \rightarrow p e^- \bar{\nu}_e) = & -2m_n \frac{G_F}{\sqrt{2}} V_{ud} \left\{ \left(1 + \frac{\alpha}{2\pi} f_{\beta_c^-}(E_e, \mu) \right) [\varphi_p^\dagger \varphi_n] [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] \right. \\ & - \tilde{\lambda} \left(1 + \frac{\alpha}{2\pi} f_{\beta_c^-}(E_e, \mu) \right) [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] - \frac{\alpha}{2\pi} g_F(E_e) [\varphi_p^\dagger \varphi_n] [\bar{u}_e (1 - \gamma^5) v_{\bar{\nu}}] \\ & + \frac{\alpha}{2\pi} \tilde{\lambda} g_F(E_e) [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \gamma^0 \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] - \frac{m_e}{2M} [\varphi_p^\dagger \varphi_n] [\bar{u}_e (1 - \gamma^5) v_{\bar{\nu}}] \\ & \left. + \frac{\tilde{\lambda}}{2M} [\varphi_p^\dagger (\vec{\sigma} \cdot \vec{k}_p) \varphi_n] [\bar{u}_e \gamma^0 (1 - \gamma^5) v_{\bar{\nu}}] - i \frac{\kappa + 1}{2M} [\varphi_p^\dagger (\vec{\sigma} \times \vec{k}_p) \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (1 - \gamma^5) v_{\bar{\nu}}] \right\}, \end{aligned} \quad (\text{D-52})$$

where $\tilde{\lambda} = \lambda(1 - E_0/2M)$ and $\vec{k}_p = -\vec{k}_e - \vec{k}$ is the proton 3-momentum in the rest frame of the neutron. The amplitude Eq.(D-52) has been used for the calculation of the electron-energy and angular distribution Eq.(6).

The radiative corrections to the rate of the continuum-state β^- -decay of the neutron, which we denote as $g_{\beta_c^-}(E_e, \mu)$, acquire an additional contribution of the electromagnetic Fierz term

$$\begin{aligned} g_{\beta_c^-}(E_e, \mu) = & f_{\beta_c^-}(E_e, \mu) + \frac{\pi}{\alpha} b_F^{(\text{em})}(E_e) \frac{m_e}{E_e} = f_{\beta_c^-}(E_e, \mu) - g_F(E_e) \frac{m_e}{E_e} = \frac{3}{2} \ln\left(\frac{m_p}{m_e}\right) - \frac{11}{8} \\ & + 2 \ln\left(\frac{\mu}{m_e}\right) \left[\frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 1 \right] + \frac{1}{\beta} L\left(\frac{2\beta}{1+\beta}\right) - \frac{1}{4\beta} \ln^2\left(\frac{1+\beta}{1-\beta}\right) + \frac{\beta}{2} \ln\left(\frac{1+\beta}{1-\beta}\right) + C_{WZ}, \end{aligned} \quad (\text{D-53})$$

where the term, proportional to $\beta/2$, is defined by

$$\frac{\beta}{2} \ln\left(\frac{1+\beta}{1-\beta}\right) = \frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - g_F(E_e) \frac{m_e}{E_e}. \quad (\text{D-54})$$

For the calculation of the contributions of the effective scalar and tensor interactions, induced by one-virtual photon exchanges, to the correlation coefficients of the electron-energy and angular distribution of the neutron β^- -decay we use the results obtained in [10] (see Eq.(28) of Ref.[10]). The corrections to the correlation coefficients from the scalar and tensor lepton-nucleon weak interactions with the left-handed neutrinos take the form

$$\begin{aligned} \delta\zeta(E_e) &= b_F \frac{m_e}{E_e} = 2 \frac{g_S - 3\lambda g_T}{1 + 3\lambda^2} \frac{m_e}{E_e}, \\ \delta a(E_e) &= 0, \\ \delta A(E_e) &= 0, \\ \delta B(E_e) &= 2 \frac{(g_T - \lambda g_S) - 2\lambda g_T}{1 + 3\lambda^2} \frac{m_e}{E_e}. \end{aligned} \quad (\text{D-55})$$

where we have kept only the linear terms in the scalar and tensor coupling constant expansions. Replacing the scalar and tensor coupling constants by their expressions, given in Eq.(D-44), we obtain

$$\begin{aligned} \delta\zeta(E_e) &= b_F \frac{m_e}{E_e} = 2 \frac{g_S - 3\lambda g_T}{1 + 3\lambda^2} \frac{m_e}{E_e} = -\frac{\alpha}{\pi} g_F \frac{m_e}{E_e}, \\ \delta a(E_e) &= 0, \\ \delta A(E_e) &= 0, \\ \delta B(E_e) &= 2 \frac{(g_T - \lambda g_S) - 2\lambda g_T}{1 + 3\lambda^2} \frac{m_e}{E_e} = -B_0 \frac{\alpha}{\pi} g_F \frac{m_e}{E_e}. \end{aligned} \quad (\text{D-56})$$

This gives the following radiative corrections to the correlation coefficients

$$\begin{aligned}\zeta(E_e) &= \left(1 + \frac{\alpha}{\pi} g_n(E_e)\right) + O(1/M), \\ a(E_e) &= a_0 \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) + O(1/M), \\ A(E_e) &= A_0 \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right) + O(1/M), \\ B(E_e) &= B_0 + O(1/M),\end{aligned}\tag{D-57}$$

where the terms $-g_F m_e/E_e$ are absorbed by the function $g_n(E_e)$ (see Eq.(D-53)). The terms of order of $O(1/M)$ are added in Eqs.(9) – (13).

The functions $g_n(E_e)$ and $f_n(E_e)$ are defined by

$$\begin{aligned}g_n(E_e) &= \lim_{\mu \rightarrow 0} [g_{\beta_c^-}^{(1)}(E_e, \mu) + g_{\beta_c^- \gamma}^{(1)}(E_e, \mu)] = \frac{3}{2} \ln\left(\frac{m_p}{m_e}\right) - \frac{3}{8} + 2 \left[\frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 1 \right] \left[\ln\left(\frac{2(E_0 - E_e)}{m_e}\right) \right. \\ &\quad \left. - \frac{3}{2} + \frac{1}{3} \frac{E_0 - E_e}{E_e} \right] + \frac{2}{\beta} L\left(\frac{2\beta}{1+\beta}\right) + \frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) \left[(1+\beta^2) + \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2} - \ln\left(\frac{1+\beta}{1-\beta}\right) \right] + C_{WZ}, \\ f_n(E_e) &= \lim_{\mu \rightarrow 0} [g_{\beta_c^-}^{(2)}(E_e, \mu) - g_{\beta_c^- \gamma}^{(1)}(E_e, \mu)] + g_F(E_e) \frac{m_e}{E_e} = \frac{2}{3} \frac{E_0 - E_e}{E_e} \left(1 + \frac{1}{8} \frac{E_0 - E_e}{E_e}\right) \frac{1 - \beta^2}{\beta^2} \\ &\quad \times \left[\frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 1 \right] - \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2} + \frac{1 - \beta^2}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right),\end{aligned}\tag{D-58}$$

where the functions $g_{\beta_c^-}^{(1)}(E_e, \mu)$ and $g_{\beta_c^- \gamma}^{(2)}(E_e, \mu)$ are given in Eq.(B-28).

The $g_n(E_e)$ and $f_n(E_e)$, multiplied by α/π , are in analytical agreement with results, obtained in [18]–[30] and [31, 32], respectively. The constant C_{WZ} , defined by the contributions of the W -boson and Z -boson exchanges and the QCD corrections [28], is equal to $C_{WZ} = 10.249$. This numerical value is obtained from the fit of the radiative corrections to the lifetime of the neutron $(\alpha/\pi)\langle g_n(E_e) \rangle = 0.03886(39)$ [1] and $(\alpha/\pi)\langle f_n(E_e) \rangle = 0.0390(8)$ [28], averaged over the phase volume of the neutron decay.

The radiative corrections $(\alpha/\pi) g_n(E_e)$ and $(\alpha/\pi) f_n(E_e)$, weighted with the electron energy spectrum density

$$\rho_{\beta_c^-}(E_e) = (E_0 - E_e)^2 \sqrt{E_e^2 - m_e^2} E_e \zeta(E_e) \frac{F(E_e, Z=1)}{f_n(E_0, Z=1)},\tag{D-59}$$

where the functions $\zeta(E_e)$ and $F(E_e, Z=1)$ are given in Eq.(7) and Eq.(5), respectively, and $f_n(E_0, Z=1)$ is the Fermi integral Eq.(40), are plotted in Fig. 1.

Finally we would like to note that having attributed the terms, proportional to $(\lambda + 1)$ to the renormalisation constants of the Fermi and axial coupling constants only, we arrive at the radiative corrections, described by the function

$$\begin{aligned}g(E_e) &= g_n(E_e) + 3\ln\left(\frac{\Lambda}{m_p}\right) + \frac{9}{8} - C_{WZ} = \\ &= 3\ln\left(\frac{\Lambda}{m_p}\right) + \frac{3}{2} \ln\left(\frac{m_p}{m_e}\right) + \frac{3}{4} + 2 \left[\frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 1 \right] \left[\ln\left(\frac{2(E_0 - E_e)}{m_e}\right) - \frac{3}{2} + \frac{1}{3} \frac{E_0 - E_e}{E_e} \right] \\ &\quad + \frac{2}{\beta} L\left(\frac{2\beta}{1+\beta}\right) + \frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) \left[(1+\beta^2) + \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2} - \ln\left(\frac{1+\beta}{1-\beta}\right) \right].\end{aligned}\tag{D-60}$$

The function $g(E_e)$, multiplied by α/π , agrees analytically with the result, calculated by Kinoshita and Sirlin [15].

Appendix E: Kinoshita-Lee-Nauenberg theorem for radiative corrections of neutron β^- -decay

As has been shown by Kinoshita and Sirlin [15], the radiative corrections to the rates of the muon decay $\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$ and of the neutron β^- -decay, taken in the limit $m_e \rightarrow 0$ and integrated over the phase volume, does not depend on the electron mass. This is so-called the Kinoshita-Lee-Nauenberg (KLN) theorem [36]. Thus, the radiative corrections to the rate of the neutron decay, described by the function $g_n(E_e)$, should obey the KLN theorem. This means that the result of the integration of the function $g_n(E_e)$, taken in the limit $m_e \rightarrow 0$, over the phase volume should not depend on m_e .

Sirlin's function $\bar{g}(E_e) = g_n(E_e) - C_{WZ}$, defining the radiative corrections, caused by one-virtual photon exchanges, to the lifetime of the neutron, takes the form [18]

$$\begin{aligned} \bar{g}(E_e) = g_n(E_e) - C_{WZ} = & \frac{3}{2} \ln\left(\frac{m_p}{m_e}\right) - \frac{3}{8} + 2 \left[\frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 1 \right] \left[\ln\left(\frac{2(E_0 - E_e)}{m_e}\right) - \frac{3}{2} + \frac{1}{3} \frac{E_0 - E_e}{E_e} \right] \\ & + \frac{2}{\beta} L\left(\frac{2\beta}{1+\beta}\right) + \frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) \left[(1+\beta^2) + \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2} - \ln\left(\frac{1+\beta}{1-\beta}\right) \right]. \end{aligned} \quad (\text{E-1})$$

Taking the limit $E_e \gg m_e$, corresponding to $\beta \rightarrow 1$, and introducing the variable $x = E_e/E_0$ we transcribe the function $g(E_e)$ into the form

$$\begin{aligned} \bar{g}(E_e) \rightarrow \bar{g}(x, m_e) = & \frac{3}{2} \ln\left(\frac{m_p}{2E_0}\right) + \frac{3}{2} \ln\left(\frac{2E_0}{m_e}\right) - \frac{3}{8} + 2 \left(\ln x + \ln\left(\frac{2E_0}{m_e}\right) - 1 \right) \left[\ln\left(\frac{1-x}{x}\right) - \frac{3}{2} + \frac{1}{3} \frac{1-x}{x} \right] \\ & + \left(\ln x + \ln\left(\frac{2E_0}{m_e}\right) \right) \left[2 + \frac{1}{12} \frac{(1-x)^2}{x^2} - 2 \left(\ln x + \ln\left(\frac{2E_0}{m_e}\right) \right) \right]. \end{aligned} \quad (\text{E-2})$$

Thus, the function $\bar{g}(x, m_e)$ is equal to

$$\begin{aligned} \bar{g}(x, m_e) = & \frac{3}{2} \ln\left(\frac{m_p}{2E_0}\right) - \frac{3}{8} - \frac{\pi^2}{3} + 2(\ln x - 1) \left[\ln\left(\frac{1-x}{x}\right) - \frac{3}{2} + \frac{1}{3} \frac{1-x}{x} \right] \\ & + \frac{1}{12} \frac{(1-x)^2}{x^2} \ln x + \ln\left(\frac{2E_0}{m_e}\right) \left[2\ln\left(\frac{1-x}{x}\right) - \frac{3}{2} + \frac{2}{3} \frac{1-x}{x} + \frac{1}{12} \frac{(1-x)^2}{x^2} \right]. \end{aligned} \quad (\text{E-3})$$

For the derivation Eq.(E-3) we have used $L(1) = -\pi^2/6$ and the approximation

$$\ln\left(\frac{1+\beta}{1-\beta}\right) \rightarrow 2\ln x + 2\ln\left(\frac{2E_0}{m_e}\right). \quad (\text{E-4})$$

According to the KLN theorem [36] (see also [15]), the function $\bar{g}(x, m_e)$, integrated over the phase volume, taken in the limit $m_e \rightarrow 0$, should not depend on m_e . We would like to emphasise that the quadratic terms $\ln^2(2E_0/m_e)$ are cancelled in the function $\bar{g}(x, m_e)$ without integration over the phase volume.

Now let check the contribution of the linear terms $\ln(2E_0/m_e)$. For this aim we have to integrate the function $g(x, m_e)$ over the phase volume. The integration of the function $g(x, m_e)$ over the phase volume with a dimensionless element $(1-x)^2 x^2 dx$, obtained at $m_e \rightarrow 0$, gives the result independent of m_e , since

$$\int_0^1 \left[2\ln\left(\frac{1-x}{x}\right) - \frac{3}{2} + \frac{2}{3} \frac{1-x}{x} + \frac{1}{12} \frac{(1-x)^2}{x^2} \right] (1-x)^2 x^2 dx = 0. \quad (\text{E-5})$$

Thus, the term $\ln(2E_0/m_e)$ vanishes. This reproduces the results, obtained in [15], and confirms the KLN theorem [36].

We would like to note that the term $-3/2$ in the integrand of Eq.(E-5), playing an important role for the vanishing of the integral, is given by $-3/2 = -3 + 3/2$, where -3 and $+3/2$ come from the energy depending part of the function $\bar{g}(E_e)$ and the term $(3/2)\ln(m_p/m_e)$, respectively.

Appendix F: Comparison with Sirlin's calculation of radiative corrections [18]

In this Appendix we compare our calculation of the radiative corrections to the continuum-state β^- -decay of the neutron, caused by one-virtual photon exchanges, with the calculation, carried out by Sirlin in his well-known paper [18].

According to Sirlin [18], the radiative corrections to the amplitude of the continuum-state β^- -decay of the neutron are defined by Eqs.(9a), (11), (14) and (19) of Ref.[18]. Now let us compare our expressions with Sirlin's ones.

Sirlin's Eq.(9a) corresponds to our amplitude \mathcal{M}_{pe} . In order to show this we rewrite the amplitude \mathcal{M}_{pe} , given by Eq.(D-2), as follows

$$\mathcal{M}_{pe}^{(\gamma)} = -\frac{\alpha}{4\pi} \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) [\bar{u}_e \gamma^\alpha \frac{1}{m_e - \hat{k}_e - \hat{q} - i0} O_\mu v_\nu] [\bar{u}_p \gamma^\beta \frac{1}{m_p - \hat{k}_p + \hat{q} - i0} W^\mu u_n], \quad (\text{F-1})$$

where $D_{\alpha\beta}(q)$ is the photon Green function

$$D_{\alpha\beta}(q) = \frac{1}{q^2 + i0} \left(g_{\alpha\beta} - \xi \frac{q_\alpha q_\beta}{q^2 + i0} \right) \quad (\text{F-2})$$

in the arbitrary gauge with a gauge parameter ξ , W^μ and O_μ are the products of the Dirac matrices defined by

$$W^\mu = \gamma^\mu(1 + \lambda\gamma^5) \quad , \quad O_\mu = \gamma_\mu(1 - \gamma^5). \quad (\text{F-3})$$

After some algebraical transformations Eq.(F-1) can be reduced to Sirlin's form

$$\mathcal{M}_{pe}^{(\gamma)} = -\frac{\alpha}{4\pi} \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) \frac{[\bar{u}_e(2k_e^\alpha + \gamma^\alpha \hat{q})O_\mu v_{\bar{\nu}}]}{q^2 + 2k_e \cdot q + i0} \bar{u}_p \left[\frac{(2k_p^\beta - q^\beta)W^\mu}{q^2 - 2k_p \cdot q + i0} + T^{\beta\mu} \right] u_n, \quad (\text{F-4})$$

where $T^{\beta\mu}$ is given by

$$T^{\beta\mu} = \frac{R^{\beta\mu}}{q^2 - 2k_p \cdot q + i0} \quad (\text{F-5})$$

and $R^{\beta\mu}$ is equal to $R^{\beta\mu} = i\sigma^{\beta\lambda} q_\lambda W^\mu$. As has been pointed out by Sirlin [18], the tensor $T^{\beta\mu}$ is obviously transverse. $q_\beta T^{\beta\mu} = 0$. Then, we propose to rewrite the amplitude Eq.(F-4) as follows

$$\mathcal{M}_{pe}^{(\gamma)} = \mathcal{M}_{pe}^{(\text{SC})} + \delta\mathcal{M}_{pe}^{(\text{SLI})}, \quad (\text{F-6})$$

where the indices (SC) and (SLI) mean ‘‘Sirlin's Corrections’’ and ‘‘Strong low-energy interactions’’. The amplitudes $\mathcal{M}_{pe}^{(\text{SC})}$ and $\delta\mathcal{M}_{pe}^{(\text{SLI})}$ are equal to

$$\begin{aligned} \mathcal{M}_{pe}^{(\text{SC})} &= -\frac{\alpha}{4\pi} \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) \frac{[\bar{u}_e(2k_e^\alpha + \gamma^\alpha \hat{q})O_\mu v_{\bar{\nu}}]}{q^2 + 2k_e \cdot q + i0} \frac{[\bar{u}_p(2k_p^\beta - q^\beta)W^\mu u_n]}{q^2 - 2k_p \cdot q + i0}, \\ \delta\mathcal{M}_{pe}^{(\text{SLI})} &= -\frac{\alpha}{4\pi} \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) \frac{[\bar{u}_e(2k_e^\alpha + \gamma^\alpha \hat{q})O_\mu v_{\bar{\nu}}]}{q^2 + 2k_e \cdot q + i0} [\bar{u}_p T^{\beta\mu} u_n]. \end{aligned} \quad (\text{F-7})$$

The contribution of Sirlin's Eq.(11) coincides with the part of the amplitude $\mathcal{M}_{ee}^{(\gamma)}$, defining renormalisation of the wave function of the electron by electromagnetic interactions. The amplitude $\mathcal{M}_{ee}^{(\gamma)}$, given by Eq.(D-2), we rewrite as

$$\mathcal{M}_{ee}^{(\gamma)} = \mathcal{M}_{ee}^{(\text{SC})} + \delta\mathcal{M}_{ee}^{(\text{SLI})}, \quad (\text{F-8})$$

where the amplitudes $\mathcal{M}_{ee}^{(\text{SC})}$ and $\delta\mathcal{M}_{ee}^{(\text{SLI})}$ are equal to

$$\begin{aligned} \mathcal{M}_{ee}^{(\text{SC})} &= -\frac{\alpha}{8\pi m_e} [\bar{u}_p W^\mu u_n] \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) \frac{[\bar{u}_e(2k_e^\alpha + \gamma^\alpha \hat{q})\hat{k}_e(2k_e^\beta + \hat{q}\gamma^\beta)(\hat{k}_e + m_e)O_\mu v_{\bar{\nu}}]}{(q^2 + 2k_e \cdot q + i0)^2} \\ \delta\mathcal{M}_{ee}^{(\text{SLI})} &= \frac{\alpha}{4\pi} [\bar{u}_p W^\mu u_n] \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) \left[\bar{u}_e \gamma^\alpha \frac{1}{m_e - \hat{k}_e - \hat{q} - i0} \gamma^\beta \frac{1}{m_e - \hat{k}_e - i0} O_\mu v_{\bar{\nu}} \right] \\ &\quad + [\bar{u}_p W^\mu u_n] \left[\bar{u}_e \left(-\delta m_e + \frac{Z_2^{(e)} - 1}{2}(m_e - \hat{k}_e) \right) \frac{1}{m_e - \hat{k}_e - i0} O_\mu v_{\bar{\nu}} \right] \\ &\quad + \frac{\alpha}{8\pi m_e} [\bar{u}_p W^\mu u_n] \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) \frac{[\bar{u}_e(2k_e^\alpha + \gamma^\alpha \hat{q})\hat{k}_e(2k_e^\beta + \hat{q}\gamma^\beta)O_\mu v_{\bar{\nu}}]}{(q^2 + 2k_e \cdot q + i0)^2}, \end{aligned} \quad (\text{F-9})$$

respectively. For the derivation of Sirlin's term $\mathcal{M}_{ee}^{(\text{SC})}$ one has to use the definition of the renormalisation constant $Z_2^{(e)} - 1$ of the electron wave function [19, 35]

$$\begin{aligned} \frac{Z_2^{(e)} - 1}{2} &= -\frac{\alpha}{8\pi} \frac{k_e^\lambda}{m_e} \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) \frac{\partial}{\partial k^\lambda} \gamma^\alpha \frac{1}{m_e - \hat{k}_e - \hat{q}} \gamma^\beta \Big|_{\hat{k}_e = m_e} = -\frac{\alpha}{8\pi} \frac{k_e^\lambda}{m_e} \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) \frac{\partial}{\partial q^\lambda} \gamma^\alpha \frac{1}{m_e - \hat{k}_e - \hat{q}} \gamma^\beta \Big|_{\hat{k}_e = m_e} = \\ &= -\frac{\alpha}{8\pi m_e} \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) \frac{(2k_e^\alpha + \gamma^\alpha \hat{q})\hat{k}_e(2k_e^\beta + \hat{q}\gamma^\beta)}{(q^2 + 2k_e \cdot q + i0)^2} \Big|_{\hat{k}_e = m_e}. \end{aligned} \quad (\text{F-10})$$

In comparison with Sirlin's expression, we have taken away the operator $(\hat{k}_e + m_e)/2m_e$, which is equal to unity at $\hat{k}_e = m_e$.

As has been pointed out by Sirlin, Eq.(14) of Ref.[18] is related to the emission and absorption of a photon by the proton. This means that Eq.(14) of Ref.[18] should be a part of the amplitude $\mathcal{M}_{pp}^{(\gamma)}$. This allows us to represent the amplitude $\mathcal{M}_{pp}^{(\gamma)}$ in the form

$$\mathcal{M}_{pp}^{(\gamma)} = \mathcal{M}_{pp}^{(\text{SC})} + \delta\mathcal{M}_{pp}^{(\text{SLI})} \quad (\text{F-11})$$

with $\mathcal{M}_{pp}^{(\text{SC})}$ and $\delta\mathcal{M}_{pp}^{(\text{SLI})}$, given by

$$\begin{aligned} \mathcal{M}_{pp}^{(\text{SC})} &= -\frac{\alpha}{8\pi} [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) \frac{(2k_p - q)^\alpha (2k_p - q)^\beta}{(q^2 - 2k_p \cdot q + i0)^2} \\ \delta\mathcal{M}_{pp}^{(\text{SLI})} &= \frac{\alpha}{4\pi} \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) [\bar{u}_p \gamma^\alpha \frac{1}{m_p - \hat{k}_p + \hat{q} - i0} \gamma^\beta \frac{1}{m_p - \hat{k}_p - i0} W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \\ &+ [\bar{u}_p \left(-\delta m_p + \frac{Z_2^{(p)} - 1}{2} (m_p - \hat{k}_p) \right) \frac{1}{m_p - \hat{k}_p - i0} W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \\ &+ \frac{\alpha}{8\pi} [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) \frac{(2k_p - q)^\alpha (2k_p - q)^\beta}{(q^2 - 2k_p \cdot q + i0)^2}, \end{aligned} \quad (\text{F-12})$$

respectively. As a result, according to Sirlin [18], the observable radiative corrections to the amplitude of the continuum-state β^- -decay of the neutron, caused by one-virtual photon exchange, are defined by the amplitude

$$\mathcal{M}_{\text{RC}}^{(\text{SC})} = \mathcal{M}_{pe}^{(\text{SC})} + \mathcal{M}_{ee}^{(\text{SC})} + \mathcal{M}_{pp}^{(\text{SC})}, \quad (\text{F-13})$$

which is gauge invariant, i.e. invariant under the gauge transformation of the photon Green function $D_{\alpha\beta}(q) \rightarrow D_{\alpha\beta}(q) + c(q^2) q_\alpha q_\beta$ with an arbitrary function $c(q^2)$, and suffers from the infrared divergences only [18]. The subscript RC means ‘‘Radiative Corrections’’. As we show below the additional contribution, described by the amplitude

$$\delta\mathcal{M}_{\text{RC}}^{(\text{SLI})} = \delta\mathcal{M}_{pe}^{(\text{SLI})} + \delta\mathcal{M}_{ee}^{(\text{SLI})} + \delta\mathcal{M}_{pp}^{(\text{SLI})}, \quad (\text{F-14})$$

does not depend on the electron energy and should be absorbed by renormalisation of the Fermi coupling constant G_F and axial coupling constant λ .

Since the amplitude $\mathcal{M}_{\text{RC}}^{(\text{SC})}$ is invariant under gauge transformations of the photon Green function $D_{\alpha\beta}(q) \rightarrow D_{\alpha\beta}(q) + c(q^2) q_\alpha q_\beta$, we may calculate it by using the Feynman gauge for the photon Green function. The calculation of the amplitude $\delta\mathcal{M}_{\text{RC}}^{(\text{SLI})}$, describing the contributions to renormalisation constants of the Fermi and axial coupling constants, can be also carried out in the Feynman gauge.

The amplitude $\mathcal{M}_{pe}^{(\text{SC})}$ can be rewritten as follows

$$\begin{aligned} \mathcal{M}_{pe}^{(\text{SC})} &= \frac{\alpha}{4\pi} [\bar{u}_p W^\mu u_n] \left\{ [\bar{u}_e O_\mu v_{\bar{\nu}}] \left[\int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} + \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} \right. \right. \\ &- \left. \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} - 4(k_e \cdot k_p) \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} \right] \\ &- \left. 2i \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} [\bar{u}_e \sigma_{\alpha\beta} q^\alpha k_p^\beta O_\mu v_{\bar{\nu}}] \right\}. \end{aligned} \quad (\text{F-15})$$

All momentum integrals in Eq.(F-15) are calculated in Appendix D by using the Pauli-Villars regularisation of the ultra-violet divergent integrals and the FPM regularisation for the infrared divergent contributions. Using the results, obtained in Appendix C and Appendix D (see Eqs.(C-15), (D-12) and (D-40)), the amplitude $\mathcal{M}_{pe}^{(\text{SC})}$ takes the form

$$\begin{aligned} \mathcal{M}_{pe}^{(\text{SC})} &= [\bar{u}_p W^\mu u_n] \frac{\alpha}{2\pi} \left\{ [\bar{u}_e O_\mu v_{\bar{\nu}}] \left[\ln\left(\frac{\Lambda}{m_e}\right) + \frac{1}{2} + \ln\left(\frac{\mu}{m_e}\right) \frac{1}{\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) + \frac{1}{\beta} L\left(\frac{2\beta}{1+\beta}\right) \right. \right. \\ &- \left. \frac{1}{4\beta} \ln^2\left(\frac{1+\beta}{1-\beta}\right) + \frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) \right] - [\bar{u}_e \gamma^0 O_\mu v_{\bar{\nu}}] \frac{\sqrt{1-\beta^2}}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) \right\}. \end{aligned} \quad (\text{F-16})$$

The amplitude $\mathcal{M}_{ee}^{(\text{SC})}$ can be transcribed into the form

$$\begin{aligned}
\mathcal{M}_{ee}^{(\text{SC})} &= -\frac{\alpha}{8\pi m_e} [\bar{u}_p W^\mu u_n] \int \frac{d^4 q}{\pi^2 i} D_{\alpha\beta}(q) \frac{[\bar{u}_e(2k_e^\alpha + \gamma^\alpha \hat{q}) \hat{k}_e(2k_e^\beta + \gamma^\beta \hat{q}) O_\mu v_{\bar{\nu}}]}{(q^2 + 2k_e \cdot q + i0)^2} \\
&= -\frac{\alpha}{4\pi m_e} [\bar{u}_p W^\mu u_n] \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{[\bar{u}_e(2m_e^3 + 2m_e \hat{q} \hat{k}_e - \hat{q} \hat{k}_e \hat{q}) O_\mu v_{\bar{\nu}}]}{(q^2 + 2k_e \cdot q + i0)^2} \rightarrow -\frac{\alpha}{4\pi m_e} [\bar{u}_p W^\mu u_n] \\
&\times \int_0^1 dx \, 2x \int \frac{d^4 q}{\pi^2 i} \left\{ \frac{[\bar{u}_e(2m_e^3 + 2m_e \hat{q} \hat{k}_e - \hat{q} \hat{k}_e \hat{q}) O_\mu v_{\bar{\nu}}]}{((q + k_e x)^2 - m_e^2 x^2 - \mu^2(1-x))^3} - \frac{[\bar{u}_e(2m_e^3 + 2m_e \hat{q} \hat{k}_e - \hat{q} \hat{k}_e \hat{q}) O_\mu v_{\bar{\nu}}]}{((q + k_e x)^2 - m_e^2 x^2 - \Lambda^2(1-x))^3} \right\} = \\
&= \frac{\alpha}{8\pi} [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \int_0^1 dx \, 2x \int \frac{d^4 q}{\pi^2} \left[\frac{4m_e^2(1-x-x^2) - q^2}{(q^2 + m_e^2 x^2 + \mu^2(1-x))^3} - \frac{m_e^2(4-4x-2x^2) - q^2}{(q^2 + m_e^2 x^2 + \Lambda^2(1-x))^3} \right] = \\
&= \frac{\alpha}{8\pi} [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \int_0^1 dx \, 2x \int \frac{d^4 q}{\pi^2} \left[\frac{m_e^2(4-4x-x^2)}{(q^2 + m_e^2 x^2 + \mu^2(1-x))^3} - \frac{m_e^2(4-4x-x^2) + \Lambda^2(1-x)}{(q^2 + m_e^2 x^2 + \Lambda^2(1-x))^3} \right. \\
&\times \left. - \frac{1}{(q^2 + m_e^2 x^2 + \mu^2(1-x))^2} + \frac{1}{(q^2 + m_e^2 x^2 + \Lambda^2(1-x))^2} \right] = \frac{\alpha}{16\pi} [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \\
&\times \int_0^1 dx \, 2x \left\{ \frac{m_e^2(4-4x-x^2)}{m_e^2 x^2 + \mu^2(1-x)} - \frac{m_e^2(4-4x-x^2) + \Lambda^2(1-x)}{m_e^2 x^2 + \Lambda^2(1-x)} - 2 \ell n \left[\frac{\Lambda^2(1-x)}{m_e^2 x^2} \right] \right\}. \tag{F-17}
\end{aligned}$$

Having integrated over x we arrive at the following expression for the amplitude $\mathcal{M}_{ee}^{(\text{SC})}$

$$\mathcal{M}_{ee}^{(\text{SC})} = [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \frac{\alpha}{2\pi} \left[-\frac{1}{2} \ell n \left(\frac{\Lambda}{m_e} \right) - \ell n \left(\frac{\mu}{m_e} \right) - \frac{9}{8} \right]. \tag{F-18}$$

In the Feynman gauge the amplitude $\mathcal{M}_{pp}^{(\text{SC})}$ takes the form

$$\begin{aligned}
\mathcal{M}_{pp}^{(\text{SC})} &= -\frac{\alpha}{8\pi} [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \int \frac{d^4 q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{4m_p^2 - 4(k_p \cdot q) + q^2}{(q^2 - 2k_p \cdot q + i0)^2} \rightarrow -\frac{\alpha}{8\pi} [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \\
&\times \int_0^1 dx \, 2x \int \frac{d^4 q}{\pi^2 i} \left[\frac{4m_p^2 - 4(k_p \cdot q) + q^2}{((q - k_p x)^2 - m_p^2 x^2 - \mu^2(1-x))^3} - \frac{4m_p^2 - 4(k_p \cdot q) + q^2}{((q - k_p x)^2 - m_p^2 x^2 - \Lambda^2(1-x))^3} \right] = \\
&= \frac{\alpha}{8\pi} [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \int_0^1 dx \, 2x \int \frac{d^4 q}{\pi^2} \left[\frac{m_p^2(4-4x+x^2) - q^2}{(q^2 + m_p^2 x^2 + \mu^2(1-x))^3} - \frac{m_p^2(4-4x+x^2) - q^2}{(q^2 + m_p^2 x^2 + \Lambda^2(1-x))^3} \right] = \\
&= \frac{\alpha}{8\pi} [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \int_0^1 dx \, 2x \int \frac{d^4 q}{\pi^2} \left[\frac{m_p^2(4-4x+2x^2)}{(q^2 + m_p^2 x^2 + \mu^2(1-x))^3} - \frac{m_p^2(4-4x+2x^2) + \Lambda^2(1-x)}{(q^2 + m_p^2 x^2 + \Lambda^2(1-x))^3} \right. \\
&- \frac{1}{(q^2 + m_p^2 x^2 + \mu^2(1-x))^2} + \frac{1}{(q^2 + m_p^2 x^2 + \Lambda^2(1-x))^2} \left. \right] = \frac{\alpha}{8\pi} [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \int_0^1 dx \, 2x \left\{ \frac{m_p^2(2-2x+x^2)}{m_p^2 x^2 + \mu^2(1-x)} \right. \\
&- \frac{1}{2} - \ell n \left[\frac{\Lambda^2(1-x)}{m_p^2 x^2} \right] \left. \right\}. \tag{F-19}
\end{aligned}$$

Using the results, obtained in Appendix D, we define $\mathcal{M}_{pp}^{(\text{SC})}$ as follows

$$\mathcal{M}_{pp}^{(\text{SC})} = [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \frac{\alpha}{2\pi} \left[-\frac{1}{2} \ell n \left(\frac{\Lambda}{m_p} \right) - \ell n \left(\frac{\mu}{m_p} \right) - \frac{3}{4} \right]. \tag{F-20}$$

After the summation of the contributions the amplitude $\mathcal{M}_{\text{RC}}^{(\text{SC})}$ takes the form

$$\begin{aligned}
\mathcal{M}_{\text{RC}}^{(\text{SC})} &= [\bar{u}_p W^\mu u_n] \frac{\alpha}{2\pi} \left\{ [\bar{u}_e O_\mu v_{\bar{\nu}}] \left[\frac{3}{2} \ell n \left(\frac{\Lambda}{m_e} \right) - \frac{11}{8} + 2 \ell n \left(\frac{\mu}{m_e} \right) \left[\frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) - 1 \right] \right. \right. \\
&+ \frac{1}{\beta} L \left(\frac{2\beta}{1+\beta} \right) - \frac{1}{4\beta} \ell n^2 \left(\frac{1+\beta}{1-\beta} \right) + \frac{1}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) \left. \right] - [\bar{u}_e \gamma^0 O_\mu v_{\bar{\nu}}] \frac{\sqrt{1-\beta^2}}{2\beta} \ell n \left(\frac{1+\beta}{1-\beta} \right) \left. \right\}. \tag{F-21}
\end{aligned}$$

Thus, the value of the constant c_S is $c_S = -11/8$. It is fixed by the requirement of gauge invariance of the observable part of the radiative corrections to the amplitude of the continuum-state β^- -decay of the neutron.

Now let us proceed to calculating the contribution of $\delta\mathcal{M}_{\text{RC}}^{(\text{SLI})}$. The term $\mathcal{M}_{pe}^{(\text{SLI})}$ is defined by

$$\begin{aligned}\delta\mathcal{M}_{pe}^{(\text{SLI})} &= -\frac{\alpha}{4\pi} \int \frac{d^4q}{\pi^2 i} D_{\alpha\beta}(q) \frac{[\bar{u}_e(2k_e^\alpha + \gamma^\alpha \hat{q})O_\mu v_{\bar{\nu}}]}{q^2 + 2k_e \cdot q + i0} [\bar{u}_p T^{\beta\mu} u_n] = \frac{\alpha}{4\pi} \left\{ [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \right. \\ &\times 3 \int \frac{d^4q}{\pi^2 i} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} + 2(\lambda + 1) [\bar{u}_p R^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \int \frac{d^4q}{\pi^2 i} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} \\ &- 2(\lambda + 1) [\bar{u}_p R^\alpha u_n] [\bar{u}_e O^\beta v_{\bar{\nu}}] \int \frac{d^4q}{\pi^2 i} \frac{q_\alpha q_\beta}{q^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} \\ &\left. - 2i \int \frac{d^4q}{\pi^2 i} \frac{1}{q^2 + i0} \frac{1}{q^2 - 2k_p \cdot q + i0} \frac{1}{q^2 + 2k_e \cdot q + i0} [\bar{u}_p \sigma_{\alpha\beta} k_e^\alpha q^\beta W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \right\},\end{aligned}\quad (\text{F-22})$$

where $R^\lambda = \gamma^\lambda(1 + \gamma^5)$. Using the results, obtained in Appendix D (see Eq.(D-12), Eq.(D-28), Eq.(D-33) and Eq.(D-38)) we obtain

$$\begin{aligned}\delta\mathcal{M}_{pe}^{(\text{SLI})} &= \frac{\alpha}{2\pi} \left[3\ell n\left(\frac{\Lambda}{m_p}\right) + \frac{3}{2} \right] [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \\ &+ \frac{\alpha}{2\pi} (\lambda + 1) \left[-2\ell n\left(\frac{\Lambda}{m_p}\right) - 1 \right] [\bar{u}_p R^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \\ &+ \frac{\alpha}{2\pi} (\lambda + 1) \left\{ g_{\alpha\beta} \left[\frac{1}{2}\ell n\left(\frac{\Lambda}{m_p}\right) + \frac{3}{4} \right] - \frac{1}{2}g_{o\alpha}g_{0\beta} \right\} [\bar{u}_p R^\alpha u_n] [\bar{u}_e O^\beta v_{\bar{\nu}}].\end{aligned}\quad (\text{F-23})$$

For $\mathcal{M}_{ee}^{(\text{SLI})}$ and $\mathcal{M}_{pp}^{(\text{SLI})}$ we calculate the following expressions

$$\begin{aligned}\mathcal{M}_{ee}^{(\text{SLI})} &= [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \left\{ \frac{Z_2^{(e)} - 1}{2} + \frac{\alpha}{2\pi} \left(-\ell n\left(\frac{\Lambda}{m_e}\right) - 2\ell n\left(\frac{\mu}{m_e}\right) - \frac{9}{4} \right) + \frac{\alpha}{2\pi} \left(\frac{1}{2}\ell n\left(\frac{\Lambda}{m_e}\right) + \ell n\left(\frac{\mu}{m_e}\right) + \frac{9}{8} \right) \right\} = \\ &= [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \left\{ \frac{Z_2^{(e)} - 1}{2} - \frac{\alpha}{2\pi} \left(\frac{1}{2}\ell n\left(\frac{\Lambda}{m_e}\right) + \ell n\left(\frac{\mu}{m_e}\right) + \frac{9}{8} \right) \right\}\end{aligned}\quad (\text{F-24})$$

and

$$\begin{aligned}\mathcal{M}_{pp}^{(\text{SLI})} &= [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \left\{ \frac{Z_2^{(p)} - 1}{2} + \frac{\alpha}{2\pi} \left(-\ell n\left(\frac{\Lambda}{m_p}\right) - 2\ell n\left(\frac{\mu}{m_p}\right) - \frac{9}{4} \right) + \frac{\alpha}{2\pi} \left(\frac{1}{2}\ell n\left(\frac{\Lambda}{m_p}\right) + \ell n\left(\frac{\mu}{m_p}\right) + \frac{3}{4} \right) \right\} = \\ &= [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \left\{ \frac{Z_2^{(e)} - 1}{2} - \frac{\alpha}{2\pi} \left(\frac{1}{2}\ell n\left(\frac{\Lambda}{m_p}\right) + \ell n\left(\frac{\mu}{m_p}\right) + \frac{3}{2} \right) \right\},\end{aligned}\quad (\text{F-25})$$

respectively. Using the definition of the renormalisation constants $Z_2^{(e)}$ and $Z_2^{(p)}$ Eq.(D-25) we obtain

$$\begin{aligned}\mathcal{M}_{ee}^{(\text{SLI})} &= 0, \\ \mathcal{M}_{pp}^{(\text{SLI})} &= [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \frac{\alpha}{2\pi} \left(-\frac{3}{8} \right).\end{aligned}\quad (\text{F-26})$$

The sum of the contributions to $\mathcal{M}_{\text{RC}}^{(\text{SLI})}$ gives one

$$\begin{aligned}\mathcal{M}_{\text{RC}}^{(\text{SLI})} &= \frac{\alpha}{2\pi} \left[3\ell n\left(\frac{\Lambda}{m_p}\right) + \frac{3}{8} \right] [\bar{u}_p W^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \\ &+ \frac{\alpha}{2\pi} (\lambda + 1) \left[-2\ell n\left(\frac{\Lambda}{m_p}\right) - 1 \right] [\bar{u}_p R^\mu u_n] [\bar{u}_e O_\mu v_{\bar{\nu}}] \\ &+ \frac{\alpha}{2\pi} (\lambda + 1) \left\{ g_{\alpha\beta} \left[\frac{1}{2}\ell n\left(\frac{\Lambda}{m_p}\right) + \frac{3}{4} \right] - \frac{1}{2}g_{o\alpha}g_{0\beta} \right\} [\bar{u}_p R^\alpha u_n] [\bar{u}_e O^\beta v_{\bar{\nu}}].\end{aligned}\quad (\text{F-27})$$

This shows that gauge non-invariant contributions do not depend on the electron energy E_e and the infrared cut-off μ and may be fully absorbed by the renormalisation constants of the Fermi and axial coupling constants (see Eq.(D-48) and Eq.(D-49)).

In order to prove the correctness of the term $-3/8$ in Eq.(D-59), giving the contribution to the observable radiative corrections to the lifetime of the neutron, we propose to sum up the contributions of $\mathcal{M}_{\text{RC}}^{(\text{SC})}$ and the terms, proportional to $[\bar{u}_p W^\mu u_n][\bar{u}_e O_\mu v_{\bar{\nu}}]$ from $\mathcal{M}_{\text{RC}}^{(\text{SLI})}$, which we denote as $\bar{\mathcal{M}}_{\text{RC}}^{(\text{SLI})}$. This gives

$$\mathcal{M}_{\text{RC}} = \mathcal{M}_{\text{RC}}^{(\text{SC})} + \bar{\mathcal{M}}_{\text{RC}}^{(\text{SLI})} = \frac{\alpha}{2\pi} \left\{ [\bar{u}_p W^\mu u_n][\bar{u}_e O_\mu v_{\bar{\nu}}] F(E_e, \mu) - [\bar{u}_p W^\mu u_n][\bar{u}_e \gamma^0 O_\mu v_{\bar{\nu}}] \frac{\sqrt{1-\beta^2}}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) \right\}, \quad (\text{F-28})$$

where the function $F(E_e, \mu)$ takes the form

$$\begin{aligned} F(E_e, \mu) = & 3\ln\left(\frac{\Lambda}{m_p}\right) + \frac{3}{2}\ln\left(\frac{m_p}{m_e}\right) - \frac{1}{4} + 2\ln\left(\frac{\mu}{m_e}\right) \left[\frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 1 \right] + \frac{1}{\beta} L\left(\frac{2\beta}{1+\beta}\right) \\ & - \frac{1}{4\beta} \ln^2\left(\frac{1+\beta}{1-\beta}\right) + \frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right). \end{aligned} \quad (\text{F-29})$$

The contribution of the radiative corrections to the lifetime of the neutron is defined by the function

$$\begin{aligned} g(E_e) = & \lim_{\mu \rightarrow 0} [F(E_e, \mu) + g_{\beta_c}^{(1)}(E_e, \mu)] = \\ = & 3\ln\left(\frac{\Lambda}{m_p}\right) + \frac{3}{2}\ln\left(\frac{m_p}{m_e}\right) + \frac{3}{4} + 2 \left[\frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) - 1 \right] \left[\ln\left(\frac{2(E_0 - E_e)}{m_e}\right) - \frac{3}{2} + \frac{1}{3} \frac{E_0 - E_e}{E_e} \right] \\ & + \frac{2}{\beta} L\left(\frac{2\beta}{1+\beta}\right) + \frac{1}{2\beta} \ln\left(\frac{1+\beta}{1-\beta}\right) \left[(1+\beta^2) + \frac{1}{12} \frac{(E_0 - E_e)^2}{E_e^2} - \ln\left(\frac{1+\beta}{1-\beta}\right) \right], \end{aligned} \quad (\text{F-30})$$

where the function $g_{\beta_c}^{(1)}(E_e, \mu)$ is given by Eq.(B-28) and describes the contribution of the radiative β^- -decay of the neutron.

The function Eq.(F-30) reproduces our result, obtained in Appendix D (see Eq.(D-60)). Then, being multiplied by (α/π) , it reproduces also the radiative corrections to the lifetime of the neutron, calculated by Kinoshita and Sirlin [15]. This corroborates the correctness of our calculation of $\mathcal{M}_{\text{RC}}^{(\text{SC})}$ and $\mathcal{M}_{\text{RC}}^{(\text{SLI})}$.

Thus, the analysis of the radiative corrections to the neutron β^- -decay, carried out in this Appendix, confirms Sirlin's assertion that the requirement of gauge invariance of the amplitude of the radiative corrections defines unambiguously the observable radiative corrections to the lifetime of the neutron. They are independent of the axial coupling constant λ , i.e. of strong low-energy interactions. The part of radiative corrections depending on the axial coupling constant λ , i.e. on strong low-energy interactions, is unobservable and absorbed by renormalisation constants of the Fermi and axial coupling constants.

Appendix G: Contribution of weak lepton–nucleon couplings beyond SM to correlation coefficients of neutron β^- -decay

In this Appendix we take into account the contributions of the weak lepton–nucleon interactions beyond the SM with left-handed and right-handed neutrinos. For this aim we use the following Hamiltonian of phenomenological weak lepton–nucleon interactions [4]–[8]

$$\begin{aligned} \mathcal{H}_W(x) = & \frac{G_F}{\sqrt{2}} V_{ud} \left\{ [\bar{\psi}_p(x) \gamma_\mu \psi_n(x)] [\bar{\psi}_e(x) \gamma^\mu (C_V + \bar{C}_V \gamma^5) \psi_{\nu_e}(x)] + [\bar{\psi}_p(x) \gamma_\mu \gamma^5 \psi_n(x)] [\bar{\psi}_e(x) \gamma^\mu (\bar{C}_A + C_A \gamma^5) \psi_{\nu_e}(x)] \right. \\ & \left. + [\bar{\psi}_p(x) \psi_n(x)] [\bar{\psi}_e(x) (C_S + \bar{C}_S \gamma^5) \psi_{\nu_e}(x)] + \frac{1}{2} [\bar{\psi}_p(x) \sigma^{\mu\nu} \gamma^5 \psi_n(x)] [\bar{\psi}_e(x) \sigma_{\mu\nu} (\bar{C}_T + C_T \gamma^5) \psi_{\nu_e}(x)] \right\}. \end{aligned} \quad (\text{G-1})$$

This is the most general form of the effective low-energy weak interactions, where the coupling constants C_i and \bar{C}_i for $i = V, A, S$ and T can be induced by the left-handed and right-handed hadronic and leptonic currents [4]–[8] and supersymmetric interactions [3] as well. They are related to the coupling constants, analogous to those which were

introduced by Herczeg [6], as follows

$$\begin{aligned}
C_V &= 1 + a_{LL}^h + a_{LR}^h + a_{RR}^h + a_{RL}^h, \\
\bar{C}_V &= -1 - a_{LL}^h - a_{LR}^h + a_{RR}^h + a_{RL}^h, \\
C_A &= -\lambda + a_{LL}^h - a_{LR}^h + a_{RR}^h - a_{RL}^h, \\
\bar{C}_A &= \lambda - a_{LL}^h + a_{LR}^h + a_{RR}^h - a_{RL}^h, \\
C_S &= A_{LL}^h + A_{LR}^h + A_{RR}^h + A_{RL}^h, \\
\bar{C}_S &= -A_{LL}^h - A_{LR}^h + A_{RR}^h + A_{RL}^h, \\
C_T &= 2(\alpha_{LL}^h + \alpha_{RR}^h), \\
\bar{C}_T &= 2(-\alpha_{LL}^h + \alpha_{RR}^h),
\end{aligned} \tag{G-2}$$

where the index h means that the coupling constants are introduced at the *hadronic* level but not at the *quark* level as it has been done by Herczeg [6]. In addition in comparison with Herczeg [6] we have taken away the common factor $G_F V_{ud}/\sqrt{2}$ and defined the coupling constants a_{LL}^h and a_{LR}^h as deviations from the coupling constants of the SM [92]. Analogous to Herczeg [6], the Hamiltonian of phenomenological lepton–nucleon weak interactions beyond the SM may be written in the following form [92]

$$\begin{aligned}
\mathcal{H}_W(x) &= \frac{G_F}{\sqrt{2}} V_{ud} \left\{ [\bar{\psi}_e(x) \gamma_\mu (1 - \gamma^5) \psi_{\nu_e}(x)] \left[\left(\frac{1 - \lambda}{2} + a_{LL}^h \right) [\bar{\psi}_p(x) \gamma^\mu (1 - \gamma^5) \psi_n(x)] \right. \right. \\
&+ \left. \left(\frac{1 + \lambda}{2} + a_{LR}^h \right) [\bar{\psi}_p(x) \gamma^\mu (1 + \gamma^5) \psi_n(x)] \right] + [\bar{\psi}_e(x) \gamma_\mu (1 + \gamma^5) \psi_{\nu_e}(x)] \left(a_{RL} [\bar{\psi}_p(x) \gamma^\mu (1 - \gamma^5) \psi_n(x)] \right. \\
&+ \left. a_{RR} [\bar{\psi}_p(x) \gamma^\mu (1 + \gamma^5) \psi_n(x)] \right) + [\bar{\psi}_e(x) (1 - \gamma^5) \psi_{\nu_e}(x)] \left(A_{LL}^h [\bar{\psi}_p(x) (1 - \gamma^5) \psi_n(x)] + A_{LR}^h [\bar{\psi}_p(x) (1 + \gamma^5) \psi_n(x)] \right) \\
&+ [\bar{\psi}_e(x) (1 + \gamma^5) \psi_{\nu_e}(x)] \left(A_{RL} [\bar{\psi}_p(x) (1 - \gamma^5) \psi_n(x)] + A_{RR} [\bar{\psi}_p(x) (1 + \gamma^5) \psi_n(x)] \right) + \frac{1}{2} \alpha_{LL}^h [\bar{\psi}_e(x) \sigma_{\mu\nu} (1 - \gamma^5) \psi_{\nu_e}(x)] \\
&\times [\bar{\psi}_p(x) \sigma^{\mu\nu} (1 - \gamma^5) \psi_n(x)] + \frac{1}{2} \alpha_{RR} [\bar{\psi}_e(x) \sigma_{\mu\nu} (1 + \gamma^5) \psi_{\nu_e}(x)] [\bar{\psi}_p(x) \sigma^{\mu\nu} (1 + \gamma^5) \psi_n(x)] \Big\}.
\end{aligned} \tag{G-3}$$

In order to express the coupling constants C_T and \bar{C}_T in terms of the coupling constants α_{LL}^h and α_{RR}^h we have used the relation $\sigma_{\mu\nu} \gamma^5 = \frac{i}{2} \varepsilon_{\mu\nu\alpha\beta} \sigma^{\alpha\beta}$ [35].

The SM is defined by the coupling constants $C_S = \bar{C}_S = C_T = \bar{C}_T = 0$, $C_V = -\bar{C}_V = 1$ and $C_A = -\bar{C}_A = -\lambda$ and the coupling constants a_{ij}^h , A_{ij}^h and α_{jj}^h for $i(j) = L$ or R are induced by interactions beyond the SM.

The amplitude of the continuum-state β^- -decay of the neutron, calculated with the Hamiltonian of weak interactions Eq.(G-1) to leading order in the large proton mass expansion, takes the form

$$\begin{aligned}
M(n \rightarrow pe^- \bar{\nu}_e) &= -2m_n \frac{G_F}{\sqrt{2}} V_{ud} \left\{ [\varphi_p^\dagger \varphi_n] [\bar{u}_e \gamma^0 (C_V + \bar{C}_V \gamma^5) v_{\bar{\nu}}] - [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \vec{\gamma} (\bar{C}_A + C_A \gamma^5) v_{\bar{\nu}}] \right. \\
&+ [\varphi_p^\dagger \varphi_n] [\bar{u}_e (C_S + \bar{C}_S \gamma^5) v_{\bar{\nu}}] + [\varphi_p^\dagger \vec{\sigma} \varphi_n] \cdot [\bar{u}_e \gamma^0 \vec{\gamma} (\bar{C}_T + C_T \gamma^5) v_{\bar{\nu}}] \Big\}.
\end{aligned} \tag{G-4}$$

The hermitian conjugate amplitude is

$$\begin{aligned}
M^\dagger(n \rightarrow pe^- \bar{\nu}_e) &= -2m_n \frac{G_F}{\sqrt{2}} V_{ud}^* \left\{ [\varphi_n^\dagger \varphi_p] [\bar{v}_{\bar{\nu}} \gamma^0 (C_V^* + \bar{C}_V^* \gamma^5) u_e] - [\varphi_n^\dagger \vec{\sigma} \varphi_p] \cdot [\bar{v}_{\bar{\nu}} \vec{\gamma} (\bar{C}_A^* + C_A^* \gamma^5) u_e] \right. \\
&+ [\varphi_n^\dagger \varphi_p] [\bar{v}_{\bar{\nu}} (C_S^* - \bar{C}_S^* \gamma^5) u_e] - [\varphi_n^\dagger \vec{\sigma} \varphi_p] \cdot [\bar{v}_{\bar{\nu}} \gamma^0 \vec{\gamma} (\bar{C}_T^* - C_T^* \gamma^5) u_e] \Big\}.
\end{aligned} \tag{G-5}$$

The correlation coefficients of the neutron β^- -decay, expressed in terms of the phenomenological coupling constants

C_j and \bar{C}_j for $j = V, A, S$ and T and calculated to leading order in the large M expansion, are equal to

$$\begin{aligned}
\xi &= |C_V|^2 + |\bar{C}_V|^2 + 3|C_A|^2 + 3|\bar{C}_A|^2 + |C_S|^2 + |\bar{C}_S|^2 + 3|C_T|^2 + 3|\bar{C}_T|^2, \\
\xi a &= |C_V|^2 + |\bar{C}_V|^2 - |C_A|^2 - |\bar{C}_A|^2 + |C_T|^2 + |\bar{C}_T|^2 - |C_S|^2 - |\bar{C}_S|^2, \\
\xi b &= 2\text{Re}\left((C_V C_S^* + \bar{C}_V \bar{C}_S^*) - 3(C_A C_T^* + \bar{C}_A \bar{C}_T^*)\right), \\
\xi A &= 2\text{Re}\left(2C_A \bar{C}_A^* - 2C_T \bar{C}_T^* - (C_V \bar{C}_A^* + \bar{C}_V C_A^*) - (C_S \bar{C}_T^* + \bar{C}_S C_T^*)\right), \\
\xi B &= -2\text{Re}\left(2C_A \bar{C}_A^* + 2C_T \bar{C}_T^* + (C_V \bar{C}_A^* + \bar{C}_V C_A^*) + (C_S \bar{C}_T^* + \bar{C}_S C_T^*)\right) \\
&\quad + 2\text{Re}\left((C_V \bar{C}_T^* + \bar{C}_V C_T^*) - (C_A \bar{C}_S^* + \bar{C}_A C_S^*) + 2(C_A \bar{C}_T^* + \bar{C}_A C_T^*)\right) \frac{m_e}{E_e}, \\
\xi D &= 2\text{Im}\left((C_V C_A^* + \bar{C}_V \bar{C}_A^*) + (C_S C_T^* + \bar{C}_S \bar{C}_T^*)\right). \tag{G-6}
\end{aligned}$$

The correlation coefficients Eq.(G-6) reproduce well the structure of the correlation coefficients, calculated in [4]–[8]. At the neglect of the $1/M$ and radiative corrections the rate of the neutron β^- -decays, expressed in terms of the coupling constants C_j and \bar{C}_j for $j = V, A, S$ and T is equal to

$$\lambda_n = \xi \frac{G_F^2 |V_{ud}|^2}{4\pi^3} f_n(E_0, Z=1) \left(1 + b \left\langle \frac{m_e}{E_e} \right\rangle\right), \tag{G-7}$$

where the Fermi integral $f_n(E_0, Z=1)$ is given by Eq.(40) at the neglect of the $1/M$ and radiative corrections. In the SM at the neglect of the $1/M$ and radiative corrections $\xi = 2(1 + 3\lambda^2)$.

The correlation coefficients of the neutron β^- -decay with the contributions of weak interactions beyond the SM Eq.(G-1), defined as the deviations from the correlation coefficients calculated in the SM with the account for the contributions of the “weak magnetism” and the radiative corrections, are equal to

$$\begin{aligned}
\zeta(E_e) &= \zeta_{\text{SM}}(E_e) + \left(b_C + b_F \frac{m_e}{E_e}\right), \\
a(E_e) &= a_{\text{SM}}(E_e) + \frac{1}{2} \frac{1}{1 + 3\lambda^2} \left((|C_V|^2 - 1) + (|\bar{C}_V|^2 - 1) - (|C_A|^2 - \lambda^2) - (|\bar{C}_A|^2 - \lambda^2)\right) - a_0 \left(b_C + b_F \frac{m_e}{E_e}\right), \\
A(E_e) &= A_{\text{SM}}(E_e) + \frac{1}{1 + 3\lambda^2} \text{Re}\left(2(C_A \bar{C}_A^* + \lambda^2) - (C_V \bar{C}_A^* + \bar{C}_V C_A^* - 2\lambda)\right) - A_0 \left(b_C + b_F \frac{m_e}{E_e}\right), \\
B(E_e) &= B_{\text{SM}}(E_e) - \frac{1}{1 + 3\lambda^2} \text{Re}\left(2(C_A \bar{C}_A^* + \lambda^2) + (C_V \bar{C}_A^* + \bar{C}_V C_A^* - 2\lambda)\right) + \frac{1}{1 + 3\lambda^2} \\
&\quad \times \text{Re}\left((C_V \bar{C}_T^* + \bar{C}_V C_T^*) - (C_A \bar{C}_S^* + \bar{C}_A C_S^*) + 2(C_A \bar{C}_T^* + \bar{C}_A C_T^*)\right) \frac{m_e}{E_e} - B_0 \left(b_C + b_F \frac{m_e}{E_e}\right), \\
D(E_e) &= D_{\text{SM}}(E_e) + \frac{1}{1 + 3\lambda^2} \text{Im}\left((C_V C_A^* - \lambda) + (\bar{C}_V \bar{C}_A^* - \lambda)\right), \tag{G-8}
\end{aligned}$$

where the correlation coefficients $\zeta_{\text{SM}}(E_e)$, $a_{\text{SM}}(E_e)$, $A_{\text{SM}}(E_e)$ and $B_{\text{SM}}(E_e)$ are given by Eqs.(10) – (13), respectively. They are induced by the SM with the contributions of the “weak magnetism”, the proton recoil and the radiative corrections. The terms in the correlation coefficients Eq.(G-8), depending on the phenomenological coupling constants C_j and \bar{C}_j for $j = V, A, S$ and T , define small corrections to the contributions, calculated in the SM. In Eq.(G-8) we have dropped the contributions of the squared values and crossing products of the scalar and tensor coupling constants.

The correlation coefficient $D_{\text{SM}}(E_e)$ has been calculated in [56]–[60]. For the electron kinetic energies $250 \text{ keV} \leq T_e \leq 455 \text{ keV}$ and the axial coupling constant $\lambda = -1, 2750$ one may estimate that $D_{\text{SM}}(E_e) \sim 10^{-5}$ [56]. Then, the coefficients b_C and b_F are defined by

$$\begin{aligned}
b_C &= \frac{1}{2} \frac{1}{1 + 3\lambda^2} \left((|C_V|^2 - 1) + (|\bar{C}_V|^2 - 1) + 3(|C_A|^2 - \lambda^2) + 3(|\bar{C}_A|^2 - \lambda^2)\right), \\
b_F &= \frac{1}{1 + 3\lambda^2} \text{Re}\left((C_V C_S^* + \bar{C}_V \bar{C}_S^*) - 3(C_A C_T^* + \bar{C}_A \bar{C}_T^*)\right), \tag{G-9}
\end{aligned}$$

where b_F is the Fierz term [1, 2, 4, 7]. For the analysis of the sensitivity of the observables of the neutron β^- -decay to contributions of interactions beyond the SM we define the corrections to the correlation coefficients Eq.(G-8), caused by new phenomenological weak interactions Eq.(G-1), in the linear approximation with respect to the Herzeg coupling constants a_{ij}^h , A_{ij}^h and α_{ij}^h for $i(j) = L, R$ Eq.(G-2) (see section IX).

The rate of the neutron β^- -decay, corrected by the contributions of interactions beyond the SM, takes the form

$$\lambda_n = (\lambda_n)_{\text{SM}} \left(1 + b_C + b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right), \quad (\text{G-10})$$

where $(\lambda_n)_{\text{SM}}$ is defined by Eq.(39) and $\langle m_e/E_e \rangle_{\text{SM}}$ is the average values, calculated with the electron-energy spectrum Eq.(D-59). The lifetime of the neutron is equal to

$$\tau_n = (\tau_n)_{\text{SM}} \left(1 - b_C - b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right). \quad (\text{G-11})$$

The contributions of the terms b_C and b_F are discussed in section IX. We show that the contribution of b_C to the lifetime of the neutron may be removed by the redefinition of the axial coupling constant and the CKM matrix element.

Appendix H: Contribution of proton recoil, caused by electron-proton final state Coulomb interaction

In this Appendix we calculate the contribution of the proton recoil, caused by the electron-proton Coulomb interaction in the final state of the decay. A velocity of a relative motion of the electron-proton pair is equal to

$$\vec{v} = \frac{\vec{k}_p}{m_p} - \frac{\vec{k}_e}{E_e}. \quad (\text{H-1})$$

The Coulomb corrections, caused by a proton recoil and calculated to order α/M , change the Fermi function Eq.(5) as follows

$$F(E_e, Z = 1) \rightarrow F(E_e, Z = 1) \left(1 - \frac{\pi\alpha}{\beta} \frac{E_e}{M} - \frac{\pi\alpha}{\beta^3} \frac{E_0 - E_e}{M} \frac{\vec{k}_e \cdot \vec{k}}{E_e E} \right). \quad (\text{H-2})$$

As a result the function $\zeta(E_e)$ acquires the following correction

$$\zeta(E_e) \rightarrow \zeta(E_e)\eta(E_e) = \zeta(E_e) \left(1 - \frac{\pi\alpha}{\beta} \frac{E_e}{M} \left(1 + \frac{1}{3} a_0 \frac{E_0 - E_e}{E_e} \right) \right). \quad (\text{H-3})$$

The function $\eta(E_e)$, averaged over the electron energy spectrum $\rho_{\beta_c^-}(E_e)$ Eq.(D-59) and equal to $\langle \eta(E_e) \rangle = 1 - 2.7 \times 10^{-5}$, defines a correction to the lifetime of the neutron of order 10^{-5} . The correlation coefficients acquire the following corrections

$$\begin{aligned} \delta a(E_e) &= a_0 \frac{2}{3} \frac{\pi\alpha}{\beta^2} \frac{(E_0 - E_e)^2}{ME_e} - \frac{\pi\alpha}{\beta^3} \frac{E_0 - E_e}{M}, & \delta A(E_e) &= -A_0 \frac{\pi\alpha}{\beta} \frac{E_e}{M}, & \delta B(E_e) &= -B_0 \frac{\pi\alpha}{\beta} \frac{E_e}{M}, \\ \delta K_n(E_e) &= -A_0 \frac{\pi\alpha}{\beta^3} \frac{E - E_e}{M}, & \delta Q_n(E_e) &= -B_0 \frac{\pi\alpha}{\beta^3} \frac{E - E_e}{M}. \end{aligned} \quad (\text{H-4})$$

In the electron energy region $250 \text{ keV} \leq T_e \leq 455 \text{ keV}$ and at $\lambda = -1.2750$ the obtained corrections are of order $\delta a(E_e) \sim -10^{-5}$, $\delta A(E_e) \sim 10^{-6}$, $\delta B(E_e) \sim -10^{-5}$, $\delta K_n(E_e) \sim 10^{-6}$ and $\delta Q_n(E_e) \sim -10^{-5}$. Thus, the corrections, induced by the proton recoil in the Coulomb electron-proton interaction in the final state of the neutron β^- -decay and calculated to order α/M , are smaller compared with corrections of order 10^{-4} of interactions beyond the SM in the experimentally used electron energy region $250 \text{ keV} \leq T_e \leq 455 \text{ keV}$.

Appendix I: Electron-proton energy distribution and proton recoil asymmetry of neutron β^- -decay

The correlation coefficient $a(E_e)$, which analytical expression is given in Eq.(11), can be hardly used directly for the experimental determination of the correlation coefficient a_0 due to impossibility to determine experimentally a correlation between the electron and antineutrino 3-momenta. For the experimental determination of the correlation coefficient a_0 one needs to measure the correlations of charged particles, i. e. the correlations between 3-momenta of the proton and electron. For the experimental analysis of the proton recoil asymmetry we need also to have the electron-proton energy and angular distribution, including the correlations between the neutron spin and the proton 3-momentum.

After the integration over the antineutrino 3-momentum the electron-proton energy-momentum and angular distribution of the neutron β^- -decay takes the form

$$\frac{d^6\lambda_{\beta^-}(E_e, \vec{k}_e, \vec{k}_p, \vec{\xi}_n)}{dE_e dk_p d\Omega_p d\Omega_{ep}} = (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{32\pi^5} |\mathcal{M}_{\beta^-}|^2 \left(1 + \frac{\alpha}{\pi} g_n(E_e)\right) F(E_e, Z=1) \delta(m_n - E_p - E_e - E) \frac{m_n}{E_p} k_e E_e k_p^2, \quad (\text{I-1})$$

where $E = |\vec{k}_p + \vec{k}_e|$, $d\Omega_p = \sin\theta_p d\theta_p d\phi_p$ is the infinitesimal element of the solid angle of the 3-momentum of the proton relative to the polarisation vector $\vec{\xi}_n$ of the neutron, i.e. $\vec{\xi}_n \cdot \vec{k}_p = P k_p \cos\theta_p$ with $P = |\vec{\xi}_n| \leq 1$. Then, $d\Omega_{ep} = \sin\theta_{ep} d\theta_{ep} d\phi_{ep}$ is the infinitesimal element of the solid angle of the correlations of the electron-proton 3-momenta $\vec{k}_e \cdot \vec{k}_p = k_e k_p \cos\theta_{ep}$ and $\vec{\xi}_n \cdot \vec{k}_e = P k_e (\cos\theta_p \cos\theta_{ep} + \sin\theta_p \sin\theta_{ep} \cos(\phi_p - \phi_{ep}))$. In the non-relativistic approximation for the proton and to next-to-leading order in the large M expansion we replace m_n/E_p by

$$\frac{m_n}{E_p} \rightarrow 1 + \frac{E_0}{M}. \quad (\text{I-2})$$

Then, $|\mathcal{M}_{\beta^-}|^2$, multiplied by the factor Eq.(I-2), is (see Eq.(A-17))

$$\begin{aligned} \left(1 + \frac{E_0}{M}\right) |\mathcal{M}_{\beta^-}|^2 = & 1 + \tilde{a}(E_e) \frac{\vec{k}_e \cdot \vec{k}}{E_e E} + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left\{ E_0 - \frac{m_e^2}{E_e} - \left(\lambda^2 + 2(\kappa + 1)\lambda\right) \frac{\vec{k} \cdot \vec{k}_p}{E} - \left(\lambda^2 - 2(\kappa + 1)\lambda\right) \frac{\vec{k}_e \cdot \vec{k}_p}{E_e} \right\} \\ & + \tilde{A}(E_e) \frac{\vec{\xi}_n \cdot \vec{k}_e}{E_e} + \tilde{B}(E_e) \frac{\vec{\xi}_n \cdot \vec{k}}{E} + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left\{ - (2\kappa + 1) \lambda (\vec{\xi}_n \cdot \vec{k}_p) + \lambda \frac{(\vec{\xi}_n \cdot \vec{k}_p)(\vec{k}_e \cdot \vec{k})}{E_e E} + \left(\lambda^2 + (\kappa + 1)\lambda + (\kappa + 1)\right) \right. \\ & \times \frac{(\vec{\xi}_n \cdot \vec{k}_e)(\vec{k} \cdot \vec{k}_p)}{E_e E} - \left.\left(\lambda^2 - (\kappa + 1)\lambda + (\kappa + 1)\right) \frac{(\vec{\xi}_n \cdot \vec{k})(\vec{k}_e \cdot \vec{k}_p)}{E_e E} \right\}, \end{aligned} \quad (\text{I-3})$$

where $\vec{k} = -\vec{k}_p - \vec{k}_e$, $E = |\vec{k}_p + \vec{k}_e|$ and we have denoted

$$\begin{aligned} \tilde{a}(E_e) &= a_0 \left(1 + \frac{1}{1 - \lambda^2} \frac{E_0}{M}\right) \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right), \\ \tilde{A}(E_e) &= A_0 \left(1 + \frac{1}{2(1 + \lambda)} \frac{E_0}{M}\right) \left(1 + \frac{\alpha}{\pi} f_n(E_e)\right), \\ \tilde{B}(E_e) &= B_0 \left(1 + \frac{1}{2(1 - \lambda)} \frac{1}{M} \left(E_0 - \frac{m_e^2}{E_e}\right)\right). \end{aligned} \quad (\text{I-4})$$

Substituting Eq.(I-3) into Eq.(I-1) we obtain the electron-proton energy-momentum and angular distribution of the neutron β^- -decay with polarised neutron and unpolarised electron and proton

$$\begin{aligned} \frac{d^6\lambda_{\beta^-}(E_e, \vec{k}_e, \vec{k}_p, \vec{\xi}_n)}{dE_e dk_p d\Omega_p d\Omega_{ep}} = & (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{32\pi^5} \left\{ 1 + \tilde{a}(E_e) \frac{\vec{k}_e \cdot \vec{k}}{E_e E} + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left(E_0 - \frac{m_e^2}{E_e} - \left(\lambda^2 + 2(\kappa + 1)\lambda\right) \frac{\vec{k} \cdot \vec{k}_p}{E} \right. \right. \\ & - \left. \left(\lambda^2 - 2(\kappa + 1)\lambda\right) \frac{\vec{k}_e \cdot \vec{k}_p}{E_e} \right) + \tilde{A}(E_e) \frac{\vec{\xi}_n \cdot \vec{k}_e}{E_e} + \tilde{B}(E_e) \frac{\vec{\xi}_n \cdot \vec{k}}{E} + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left(- (2\kappa + 1) \lambda (\vec{\xi}_n \cdot \vec{k}_p) + \lambda \frac{(\vec{\xi}_n \cdot \vec{k}_p)(\vec{k}_e \cdot \vec{k})}{E_e E} \right. \\ & + \left. \left(\lambda^2 + (\kappa + 1)\lambda + (\kappa + 1)\right) \frac{(\vec{\xi}_n \cdot \vec{k}_e)(\vec{k} \cdot \vec{k}_p)}{E_e E} - \left.\left(\lambda^2 - (\kappa + 1)\lambda + (\kappa + 1)\right) \frac{(\vec{\xi}_n \cdot \vec{k})(\vec{k}_e \cdot \vec{k}_p)}{E_e E} \right) \right\} \\ & \times \left(1 + \frac{\alpha}{\pi} g_n(E_e)\right) F(E_e, Z=1) \delta(m_n - E_p - E_e - E) k_e E_e k_p^2, \end{aligned} \quad (\text{I-5})$$

Now let us analyse the case of unpolarised neutrons. Integrating over the solid angles $d\Omega_p$ and $d\Omega_{ep}$ we obtain the electron-proton energy spectrum

$$\frac{d^2\lambda_{\beta^-}(E_e, T_p)}{dE_e dT_p} = M (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{4\pi^3} a(E_e, T_p) \left(1 + \frac{\alpha}{\pi} g_n(E_e)\right) F(E_e, Z=1) E_e, \quad (\text{I-6})$$

where $T_p = k_p^2/2M$ is the kinetic energy of the proton. The electron-proton energy distribution $a(E_e, T_p)$ takes the form

$$a(E_e, T_p) = \zeta_1(E_e, T_p) + \tilde{a}(E_e) \zeta_2(E_e, T_p). \quad (\text{I-7})$$

The functions $\zeta_1(E_e, T_p)$ and $\zeta_2(E_e, T_p)$ are determined by the integrals over $\cos \theta_{ep} = \vec{k}_e \cdot \vec{k}_p / k_e k_p$

$$\begin{aligned} \zeta_1(E_e, T_p) = k_e k_p \int_{-1}^{+1} \left\{ 1 + \frac{1}{M} \frac{1}{1+3\lambda^2} \left[E_0 - \frac{m_e^2}{E_e} + \left(\lambda^2 + 2(\kappa+1)\lambda \right) \frac{k_p^2 + \vec{k}_e \cdot \vec{k}_p}{|\vec{k}_p + \vec{k}_e|} - \left(\lambda^2 - 2(\kappa+1)\lambda \right) \frac{\vec{k}_e \cdot \vec{k}_p}{E_e} \right] \right\} \\ \times \delta(m_n - E_p - E_e - |\vec{k}_p + \vec{k}_e|) d \cos \theta_{ep} \end{aligned} \quad (\text{I-8})$$

and

$$\zeta_2(E_e, T_p) = -k_e k_p \int_{-1}^{+1} \frac{k_e^2 + \vec{k}_e \cdot \vec{k}_p}{E_e |\vec{k}_p + \vec{k}_e|} \delta(m_n - E_p - E_e - |\vec{k}_p + \vec{k}_e|) d \cos \theta_{ep}. \quad (\text{I-9})$$

The integration over $\cos \theta_{ep}$ we perform by making a change of variables $E = |\vec{k}_p + \vec{k}_e|$. This gives

$$\begin{aligned} \zeta_1(E_e, T_p) &= \left\{ E + \frac{1}{M} \frac{1}{1+3\lambda^2} \left[\left(E_0 - \frac{m_e^2}{E_e} \right) E + \frac{1}{2} \left(\lambda^2 + 2(\kappa+1)\lambda \right) (E^2 + k_p^2 - k_e^2) \right. \right. \\ &\quad \left. \left. - \frac{1}{2} \frac{E}{E_e} \left(\lambda^2 - 2(\kappa+1)\lambda \right) (E^2 - k_p^2 - k_e^2) \right] \right\}, \\ \zeta_2(E_e, T_p) &= -\frac{1}{2} \frac{1}{E_e} (E^2 - k_p^2 + k_e^2) \end{aligned} \quad (\text{I-10})$$

at $E = m_n - E_p - E_e$. The energy region of the definition of the distribution $a(E_e, T_p)$ is given by [93]

$$\begin{aligned} 0 \leq T_p \leq (T_p)_{\max} &= \frac{(m_n - m_p)^2 - m_e^2}{2m_n} = \frac{E_0^2 - m_e^2}{2M} = 0.751 \text{ keV}, \\ (E_e)_{\min} \leq E_e \leq (E_e)_{\max}, \end{aligned} \quad (\text{I-11})$$

where $(E_e)_{\min/\max}$ are equal to

$$(E_e)_{\min/\max} = \frac{(m_n - E_p \mp k_p)^2 + m_e^2}{2(m_n - E_n \mp k_p)} = \frac{(E_0 \mp \sqrt{2MT_p})^2 + m_e^2}{2(E_0 \mp \sqrt{2MT_p})} + \frac{1}{2} \left(1 - \frac{m_e^2}{(E_0 \mp \sqrt{2MT_p})^2} \right) ((T_p)_{\max} - T_p). \quad (\text{I-12})$$

Here we have kept the next-to-leading terms in the large M expansion and used the relation

$$m_n - m_p = E_0 + \frac{E_0^2 - m_e^2}{2M} = E_0 + (T_p)_{\max}. \quad (\text{I-13})$$

To next-to-leading order in the large M expansion the variable E is defined by

$$E = m_n - E_p - E_e = E_0 - E_e + ((T_p)_{\max} - T_p). \quad (\text{I-14})$$

The functions $\zeta_1(E_e, T_p)$ and $\zeta_2(E_e, T_p)$, determined to order $1/M$, are equal to

$$\begin{aligned} \zeta_1(E_e, T_p) &= (E_0 - E_e) + ((T_p)_{\max} - T_p) + \frac{1}{M} \frac{1}{1+3\lambda^2} \left[\left(E_0 - \frac{m_e^2}{E_e} \right) (E_0 - E_e) \right. \\ &\quad \left. - \frac{1}{2} \frac{E_0}{E_e} \left((E_0 - E_e)^2 + E_e^2 - m_e^2 - 2MT_p \right) + \frac{1}{2} \left(\lambda^2 + 2(\kappa+1)\lambda \right) \left((E_0 - E_e)^2 - E_e^2 + m_e^2 - 2MT_p \right) \right. \\ &\quad \left. - \frac{1}{2} \left(\lambda^2 - 2(\kappa+1)\lambda \right) \frac{E_0 - E_e}{E_e} \left((E_0 - E_e)^2 - E_e^2 + m_e^2 - 2MT_p \right) \right], \\ \zeta_2(E_e, T_p) &= -\frac{1}{2} \frac{1}{E_e} \left((E_0 - E_e)^2 + E_e^2 - m_e^2 - 2MT_p + 2(E_0 - E_e)((T_p)_{\max} - T_p) \right). \end{aligned} \quad (\text{I-15})$$

Integrating the electron-proton energy spectrum Eq.(I-6) over the electron energy we obtain the proton-energy spectrum

$$\frac{d\lambda_{\beta_e^-}(T_p)}{dT_p} = M (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{4\pi^3} a(T_p), \quad (\text{I-16})$$

where $a(T_p)$ is defined by

$$a(T_p) = g_1(T_p) + a_0 \left(1 + \frac{1}{1 - \lambda^2} \frac{E_0}{M} \right) g_2(T_p) \quad (\text{I-17})$$

with the functions $g_1(T_p)$ and $g_2(T_p)$, given by the integrals

$$\begin{aligned} g_1(T_p) &= \int_{(E_e)_{\min}}^{(E_e)_{\max}} \zeta_1(E_e, T_p) \left(1 + \frac{\alpha}{\pi} g_n(E_e) \right) F(E_e, Z=1) E_e dE_e, \\ g_2(T_p) &= \int_{(E_e)_{\min}}^{(E_e)_{\max}} \zeta_2(E_e, T_p) \left(1 + \frac{\alpha}{\pi} g_n(E_e) + \frac{\alpha}{\pi} f_n(E_e) \right) F(E_e, Z=1) E_e dE_e. \end{aligned} \quad (\text{I-18})$$

For the calculation of the proton recoil asymmetry C , first, we have to integrate over the azimuthal angles. After the integration we arrive at the expression

$$\begin{aligned} \frac{d^4 \lambda_{\beta_c^-}(E_e, k_p, \theta_p, \theta_{ep}, P)}{dE_e dk_p d \cos \theta_p d \cos \theta_{ep}} &= (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{8\pi^3} \left\{ 1 - \tilde{a}(E_e) \frac{k_e^2 + k_e k_p \cos \theta_{ep}}{E_e E} + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \right. \\ &\times \left(E_0 - \frac{m_e^2}{E_e} + \left(\lambda^2 + 2(\kappa + 1)\lambda \right) \frac{k_p^2 + k_e k_p \cos \theta_{ep}}{E} - \left(\lambda^2 - 2(\kappa + 1)\lambda \right) \frac{k_e k_p \cos \theta_{ep}}{E_e} \right) \\ &+ P \cos \theta_p \left[\tilde{A}(E_e) \frac{k_e \cos \theta_{ep}}{E_e} - \tilde{B}(E_e) \frac{k_e \cos \theta_{ep} + k_p}{E} + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left(\left(- (2\kappa + 1)\lambda - \lambda \frac{k_e^2 + k_e k_p \cos \theta_{ep}}{E_e E} \right) k_p \right. \right. \\ &\left. \left. - 2(\kappa + 1)\lambda \frac{k_e k_p \cos \theta_{ep}}{E_e E} (k_p + k_e \cos \theta_{ep}) \right) \right] \left. \right\} \left(1 + \frac{\alpha}{\pi} g_n(E_e) \right) F(E_e, Z=1) \delta(m_n - E_p - E_e - E) k_e E_e k_p^2. \end{aligned} \quad (\text{I-19})$$

Having integrated over $\cos \theta_{ep}$ we obtain the following result

$$\begin{aligned} \frac{d^3 \lambda_{\beta_c^-}(E_e, k_p, \theta_p, P)}{dE_e dk_p d \cos \theta_p} &= (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{16\pi^3} \left\{ 2(E_0 - E_e) E_e k_p - \tilde{a}(E_e) ((E_0 - E_e)^2 - k_p^2 + k_e^2) k_p + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \right. \\ &\times \left(2(E_0 E_e - m_e^2)(E_0 - E_e) k_p + (1 + 3\lambda^2) E_e (E_0^2 - m_e^2 - k_p^2) - (1 - \lambda^2)(E_0 - E_e)(E_0^2 - m_e^2 - k_p^2) k_p \right. \\ &\left. + (\lambda^2 + 2(\kappa + 1)\lambda) E_e ((E_0 - E_e)^2 + k_p^2 - k_e^2) k_p - (\lambda^2 - 2(\kappa + 1)\lambda) (E_0 - E_e) ((E_0 - E_e)^2 - k_p^2 - k_e^2) k_p \right) \\ &+ P \cos \theta_p \left[\tilde{A}(E_e) (E_0 - E_e) ((E_0 - E_e)^2 - k_p^2 - k_e^2) - \tilde{B}(E_e) E_e ((E_0 - E_e)^2 + k_p^2 - k_e^2) + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \right. \\ &\times \left(- (2\kappa + 1)\lambda 2(E_0 - E_e) E_e k_p^2 - \lambda ((E_0 - E_e)^2 - k_p^2 + k_e^2) k_p^2 - (\kappa + 1)\lambda (((E_0 - E_e)^2 - k_e^2)^2 - k_p^4) \right. \\ &\left. \left. - \lambda(1 + \lambda) (3(E_0 - E_e)^2 - k_p^2 - k_e^2)(E_0^2 - m_e^2 - k_p^2) + \lambda(1 - \lambda) 2(E_0 - E_e) E_e (E_0^2 - m_e^2 - k_p^2) \right) \right] \left. \right\} \\ &\times \left(1 + \frac{\alpha}{\pi} g_n(E_e) \right) F(E_e, Z=1), \end{aligned} \quad (\text{I-20})$$

where we have kept the terms of order $1/M$ and the radiative corrections of order α/π . The integration over the electron energy E_e and the proton momentum k_p gives one

$$\begin{aligned} \frac{d\lambda_{\beta_c^-}(\theta_p, P)}{d \cos \theta_p} &= (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{16\pi^3} \left\{ X_1 + \frac{\alpha}{\pi} X_2 + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left[X_3 + (1 + 3\lambda^2) (X_4 + Y_1) - (1 - \lambda^2) (X_5 + Y_2) \right. \right. \\ &+ \left(\lambda^2 + 2(\kappa + 1)\lambda \right) X_6 - \left(\lambda^2 - 2(\kappa + 1)\lambda \right) X_7 \left. \right] + P \cos \theta_p \left[- (A_0 + B_0) X_8 + A_0 X_9 + A_0 \frac{\alpha}{\pi} X_{10} - B_0 \frac{\alpha}{\pi} X_{11} \right. \\ &\left. \left. + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left(\lambda X_{12} - (\kappa + 1)\lambda X_{13} - (2\kappa + 1)\lambda X_{14} - \lambda(1 + \lambda) (X_{15} + Y_3) + \lambda(1 - \lambda) (X_{16} + Y_4) \right) \right] \right\}, \end{aligned} \quad (\text{I-21})$$

where we have denoted

$$\begin{aligned}
X_1 &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} 2(E_0 - E_e) E_e k_p F(E_e, Z = 1) dE_e dk_p = 0.235044 \text{ MeV}^5, \\
X_2 &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} 2(E_0 - E_e) E_e k_p g_n(E_e) F(E_e, Z = 1) dE_e dk_p = 3.932276 \text{ MeV}^5, \\
X_3 &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} 2(E_0 E_e - m_e^2)(E_0 - E_e) k_p F(E_e, Z = 1) dE_e dk_p = 0.225101 \text{ MeV}^5, \\
X_4 &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} E_e (E_0^2 - m_e^2 - k_p^2) F(E_e, Z = 1) dE_e dk_p = 0.308674 \text{ MeV}^5, \\
X_5 &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} (E_0 - E_e)(E_0^2 - m_e^2 - k_p^2) k_p F(E_e, Z = 1) dE_e dk_p = 0.112937 \text{ MeV}^5, \\
X_6 &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} E_e ((E_0 - E_e)^2 + k_p^2 - k_e^2) k_p F(E_e, Z = 1) dE_e dk_p = 0.112937 \text{ MeV}^5, \\
X_7 &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} (E_0 - E_e)((E_0 - E_e)^2 - k_p^2 - k_e^2) k_p F(E_e, Z = 1) dE_e dk_p = -0.112164 \text{ MeV}^5, \\
X_8 &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} E_e ((E_0 - E_e)^2 + k_p^2 - k_e^2) F(E_e, Z = 1) dE_e dk_p = 0.129702 \text{ MeV}^5, \\
X_9 &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} \left[(E_0 - E_e)((E_0 - E_e)^2 - k_p^2 - k_e^2) + E_e((E_0 - E_e)^2 + k_p^2 - k_e^2) \right] F(E_e, Z = 1) dE_e dk_p = \\
&= 0.001061 \text{ MeV}^5, \\
X_{10} &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} (E_0 - E_e)((E_0 - E_e)^2 - k_p^2 - k_e^2)(g_n(E_e) + f_n(E_e)) F(E_e, Z = 1) dE_e dk_p = \\
&= -2.157114 \text{ MeV}^5, \\
X_{11} &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} E_e ((E_0 - E_e)^2 + k_p^2 - k_e^2) g_n(E_e) F(E_e, Z = 1) dE_e dk_p = 2.244201 \text{ MeV}^5, \\
X_{12} &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} [(E_0 E_e - m_e^2)((E_0 - E_e)^2 + k_p^2 - k_e^2) - E_0(E_0 - E_e)((E_0 - E_e)^2 - k_p^2 - k_e^2) \\
&\quad - ((E_0 - E_e)^2 - k_p^2 + k_e^2) k_p^2] F(E_e, Z = 1) dE_e dk_p = 0.306001 \text{ MeV}^5, \\
X_{13} &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} (((E_0 - E_e)^2 - k_e^2)^2 - k_p^4) F(E_e, Z = 1) dE_e dk_p = -0.119183 \text{ MeV}^5, \\
X_{14} &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} 2(E_0 - E_e) E_e k_p^2 F(E_e, Z = 1) dE_e dk_p = 0.186818 \text{ MeV}^5, \\
X_{15} &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} (3(E_0 - E_e)^2 - k_p^2 - k_e^2)(E_0^2 - m_e^2 - k_p^2) F(E_e, Z = 1) dE_e dk_p = -0.002705 \text{ MeV}^5, \\
X_{16} &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} 2(E_0 - E_e) E_e (E_0^2 - m_e^2 - k_p^2) F(E_e, Z = 1) dE_e dk_p = 0.291295 \text{ MeV}^5, \\
X_{17} &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} \frac{m_e}{E_e} ((E_0 - E_e)^2 - k_p^2 - k_e^2) F(E_e, Z = 1) dE_e dk_p = -0.211978 \text{ MeV}^5, \\
X_{18} &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}}^{(E_e)_{\max}} m_e ((E_0 - E_e)^2 + k_p^2 - k_e^2) F(E_e, Z = 1) dE_e dk_p = 0.093562 \text{ MeV}^5. \tag{I-22}
\end{aligned}$$

The factors X_{17} and X_{18} are related to contributions of interactions beyond the SM (see section IX). The factors X_j

for $j = 1, \dots, 16$ are calculated for the limits of the integration over E_e and k_p equal to

$$\begin{aligned} 0 &\leq k_p \leq (k_p)_{\max} = \sqrt{E_0^2 - m_e^2}, \\ (E_e)_{\min} &= \frac{(E_0 - k_p)^2 + m_e^2}{2(E_0 - k_p)} \leq E_e \leq (E_e)_{\max} = \frac{(E_0 + k_p)^2 + m_e^2}{2(E_0 + k_p)}. \end{aligned} \quad (\text{I-23})$$

The factors Y_j for $j = 1, 2, 3, 4$ are defined by the contributions of the $1/M$ corrections to the limits of the integration over E_e and k_p . The $1/M$ corrections to the limits of the integration over E_e are given in Eq.(I-12). The limits of the integration over k_p have no $1/M$ corrections

$$(k_p)_{\max} = \sqrt{\frac{m_p}{m_n}((m_n - m_p)^2 - m_e^2)} = \sqrt{E_0^2 - m_e^2}. \quad (\text{I-24})$$

The account for the $1/M$ corrections to the limits of the integration over E_e can be carried out by the formula

$$\begin{aligned} \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min} + \Delta E_-(k_p)}^{(E_e)_{\max} + \Delta E_+(k_p)} \Phi(E_e, k_p) dE_e dk_p &= \int_0^{(k_p)_{\max}} \int_{(E_e)_{\min}(k_p)}^{(E_e)_{\max}(k_p)} \Phi(E_e, k_p) dE_e dk_p \\ &+ \int_0^{(k_p)_{\max}} \left[\Phi((E_e)_{\max}, k_p) \Delta E_+(k_p) - \Phi((E_e)_{\min}, k_p) \Delta E_-(k_p) \right] dk_p \end{aligned} \quad (\text{I-25})$$

The factors Y_j for $j = 1, 2, 3, 4$ are defined by

$$\begin{aligned} Y_1 &= \frac{1}{2} \int_0^{(k_p)_{\max}} \left[(E_0 - (E_e)_{\max}) (E_e)_{\max} \left(1 - \frac{m_e^2}{(E_0 + k_p)^2} \right) F((E_e)_{\max}, Z = 1) \right. \\ &\quad \left. - (E_0 - (E_e)_{\min}) (E_e)_{\min} \left(1 - \frac{m_e^2}{(E_0 - k_p)^2} \right) F((E_e)_{\min}, Z = 1) \right] (E_0^2 - m_e^2 - k_p^2) k_p dk_p = 0.077953 \text{ MeV}^5, \\ Y_2 &= \frac{1}{4} \int_0^{(k_p)_{\max}} \left[\left((E_0 - (E_e)_{\max})^2 - k_p^2 + (E_e)_{\max}^2 - m_e^2 \right) \left(1 - \frac{m_e^2}{(E_0 + k_p)^2} \right) \right. \\ &\quad \times F((E_e)_{\max}, Z = 1) - \left((E_0 - (E_e)_{\min})^2 - k_p^2 + (E_e)_{\min}^2 - m_e^2 \right) \left(1 - \frac{m_e^2}{(E_0 - k_p)^2} \right) \\ &\quad \left. \times F((E_e)_{\min}, Z = 1) \right] (E_0^2 - m_e^2 - k_p^2) k_p dk_p = -0.000773 \text{ MeV}^5, \\ Y_3 &= \frac{1}{2} \int_0^{(k_p)_{\max}} \left[(E_0 - (E_e)_{\max}) \left((E_0 - (E_e)_{\max})^2 - k_p^2 - (E_e)_{\max}^2 + m_e^2 \right) \left(1 - \frac{m_e^2}{(E_0 + k_p)^2} \right) \right. \\ &\quad \times F((E_e)_{\max}, Z = 1) - (E_0 - (E_e)_{\min}) \left((E_0 - (E_e)_{\min})^2 - k_p^2 - (E_e)_{\min}^2 + m_e^2 \right) \left(1 - \frac{m_e^2}{(E_0 - k_p)^2} \right) \\ &\quad \left. \times F((E_e)_{\min}, Z = 1) \right] (E_0^2 - m_e^2 - k_p^2) dk_p = -0.134228 \text{ MeV}^5, \\ Y_4 &= \frac{1}{2} \int_0^{(k_p)_{\max}} \left[(E_e)_{\max} \left((E_0 - (E_e)_{\max})^2 + k_p^2 - (E_e)_{\max}^2 + m_e^2 \right) \left(1 - \frac{m_e^2}{(E_0 + k_p)^2} \right) \right. \\ &\quad \times F((E_e)_{\max}, Z = 1) - (E_e)_{\min} \left((E_0 - (E_e)_{\min})^2 + k_p^2 - (E_e)_{\min}^2 + m_e^2 \right) \left(1 - \frac{m_e^2}{(E_0 - k_p)^2} \right) \\ &\quad \left. \times F((E_e)_{\min}, Z = 1) \right] (E_0^2 - m_e^2 - k_p^2) dk_p = -0.143549 \text{ MeV}^5. \end{aligned} \quad (\text{I-26})$$

The correctness of our calculation of the electron-proton energy distribution $a(E_e, T_p)$ we may verify by calculating the lifetime of the neutron. Having integrated Eq.(I-21) over $\cos \theta_p$ we obtain the rate of the neutron β^- -decay

$$\begin{aligned} (\lambda_{\beta_c^-})_{\text{SM}} &= (1 + 3\lambda^2) \frac{G_F^2 |V_{ud}|^2}{8\pi^3} \left\{ X_1 + \frac{\alpha}{\pi} X_2 + \frac{1}{M} \frac{1}{1 + 3\lambda^2} \left[X_3 + (1 + 3\lambda^2) (X_4 + Y_1) - (1 - \lambda^2) (X_5 + Y_2) \right. \right. \\ &\quad \left. \left. + (\lambda^2 + 2(\kappa + 1)\lambda) X_6 - (\lambda^2 - 2(\kappa + 1)\lambda) X_7 \right] \right\}, \end{aligned} \quad (\text{I-27})$$

related to the lifetime as $(\lambda_{\beta_c^-})_{\text{SM}}^{-1} = (\tau_n)_{\text{SM}}$. The numerical factors X_j and Y_k for $j = 1, \dots, 7$ and $k = 1, 2$ define the lifetime of the neutron with the account for the radiative ($X_2 \neq 0$) and the $1/M$ ($X_j \neq 0$ for $j = 3, \dots, 7$ and $Y_k \neq 0$ for $k = 1, 2$) corrections.

Let us consider three possibilities: 1) $X_j = 0$ for $j = 2, \dots, 7$ and $Y_k = 0$ for $k = 1, 2$ (i.e. without radiative and $1/M$ corrections), 2) $X_k \neq 0$ for $k = 1, 2$ and $X_j = 0$ for $j = 3, \dots, 7$ and $Y_k = 0$ for $k = 1, 2$ (i.e. with radiative but without $1/M$ corrections) and 3) $X_k \neq 0$ for $k = 1, \dots, 7$ and $Y_k \neq 0$ for $k = 1, 2$ (with radiative and $1/M$ corrections). For these cases we obtain 1) $(\tau_n)_{\text{SM}} = 915.3 \text{ s}$, 2) $(\tau_n)_{\text{SM}} = 881.0 \text{ s}$ and 3) $(\tau_n)_{\text{SM}} = 879.6 \text{ s}$, respectively. These results agree well with the lifetimes of the neutron, adduced in Table I (see section X).

The correlation of the neutron spin and the proton 3-momentum is described by the angular distribution [71]

$$4\pi \frac{dW(\theta_p)}{d\Omega_p} = 1 + 2PC \cos \theta_p, \quad (\text{I-28})$$

where P is a neutron polarisation. The correlation coefficient C is defined by

$$\begin{aligned} C = & -\frac{1}{2} \frac{X_8}{X_1} (A_0 + B_0) + \frac{1}{2} \frac{X_9}{X_1} A_0 + \frac{\alpha}{\pi} \frac{1}{2} \frac{X_{10}}{X_1} A_0 - \frac{\alpha}{\pi} \frac{1}{2} \frac{X_{11}}{X_1} B_0 + \frac{1}{M} \frac{1}{1+3\lambda^2} \left(\lambda \frac{1}{2} \frac{X_{12}}{X_1} - (\kappa+1) \lambda \frac{1}{2} \frac{X_{13}}{X_1} \right. \\ & - (2\kappa+1) \lambda \frac{1}{2} \frac{X_{14}}{X_1} - \lambda(1+\lambda) \frac{1}{2} \frac{X_{15}+Y_3}{X_1} + \lambda(1-\lambda) \frac{1}{2} \frac{X_{16}+Y_4}{X_1} \Big) - \frac{\alpha}{\pi} \frac{1}{2} \frac{X_2}{X_1} - \frac{1}{M} \frac{1}{1+3\lambda^2} \left(\frac{1}{2} \frac{X_3}{X_1} + (1+3\lambda^2) \right. \\ & \times \frac{1}{2} \frac{X_4+Y_1}{X_1} - (1-\lambda^2) \frac{1}{2} \frac{X_5+Y_2}{X_1} + (\lambda^2+2(\kappa+1)\lambda) \frac{1}{2} \frac{X_6}{X_1} - (\lambda^2-2(\kappa+1)\lambda) \frac{1}{2} \frac{X_7}{X_1} \Big), \end{aligned} \quad (\text{I-29})$$

where the factor $x_C = X_8/2X_1 = 0.27491$ agrees well with the factor $x_C = 0.27594$, calculated by Glück [72]. The appearance of the term $A_0 X_9/2X_1$ is related to the deviation of the Fermi function $F(E_e, Z=1)$ from unity.

The contributions of interactions beyond the SM (see Appendix G) changes the electron-proton energy distribution $a(E_e, T_p)$ as follows

$$\begin{aligned} a(E_e, T_p) = & \left(1 + b_C + b_F \frac{m_e}{E_e} \right) \zeta_1(E_e, T_p) + \left[a_0 + \frac{1}{2} \frac{1}{1+3\lambda^2} \left((|C_V|^2 - 1) + (|\bar{C}_V|^2 - 1) - (|C_A|^2 - \lambda^2) \right. \right. \\ & \left. \left. - (|\bar{C}_A|^2 - \lambda^2) \right) \right] \left(1 + \frac{1}{1-\lambda^2} \frac{E_0}{M} \right) \left(1 + \frac{\alpha}{\pi} f_n(E_e) \right) \zeta_2(E_e, T_p). \end{aligned} \quad (\text{I-30})$$

Multiplying $a(E_e, T_p)$ by the lifetime of the neutron τ_n (see Eq.(G-11)) we obtain

$$\begin{aligned} \tau_n a(E_e, T_p) = & (\tau_n)_{\text{SM}} \left\{ \left(1 + b_F \frac{m_e}{E_e} - b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right) \zeta_1(E_e, T_p) + \left[a_0 + \frac{1}{2} \frac{1}{1+3\lambda^2} \left((|C_V|^2 - 1) + (|\bar{C}_V|^2 - 1) \right. \right. \right. \\ & \left. \left. - (|C_A|^2 - \lambda^2) - (|\bar{C}_A|^2 - \lambda^2) \right) - a_0 \left(b_C + b_F \left\langle \frac{m_e}{E_e} \right\rangle_{\text{SM}} \right) \right] \left(1 + \frac{1}{1-\lambda^2} \frac{E_0}{M} \right) \left(1 + \frac{\alpha}{\pi} f_n(E_e) \right) \zeta_2(E_e, T_p) \Big\}. \end{aligned} \quad (\text{I-31})$$

This allows to reduce the analysis of contributions of vector and axial-vector interactions beyond the SM to the electron-proton energy distribution to the analysis of these contributions to the axial coupling constant λ (see section VI and section IX). Contributions of interactions beyond the SM to the proton recoil asymmetry C are given in section IX.

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